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A class of β-perfect graphs

Celina M. Herrera de Figueiredo[a,1], Kristina Vušković[b,*,2]

[a] *Instituto de Matemática and COPPE, Universidade do Rio de Janeiro, Caixa Postal 68530, 21945-970 Rio de Janeiro, RJ, Brazil*

[b] *Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA*

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Abstract

Consider the following total order: order the vertices by repeatedly removing a vertex of minimum degree in the subgraph of vertices not yet chosen and placing it after all the remaining vertices but before all the vertices already removed. For which graphs the greedy algorithm on this order gives an optimum vertex-coloring? Markossian, Gasparian and Reed introduced the class of β-perfect graphs. These graphs admit such a greedy vertex-coloring algorithm. The recognition of β-perfect graphs is open. We define a subclass of β-perfect graphs, that can be recognized in polynomial time, by considering the class of graphs with no even hole, no short-chorded cycle on six vertices, and no diamond. In particular, we make use of the following properties: no minimal β-imperfect graph contains a simplicial vertex, a minimal β-imperfect graph which is not an even hole contains no vertex of degree 2.

1. Introduction

Let $G=(V(G),E(G))$ be a graph without loops or multiple edges. We denote the chromatic number of G by $\chi(G)$. We let δ_G be the minimum degree of a vertex in G. Consider the following total order on $V(G)$: order the vertices by repeatedly removing a vertex of minimum degree in the subgraph of vertices not yet chosen and placing it after all the remaining vertices but before all the vertices already removed. Coloring

* Corresponding author.

E-mail address: kristina@ms.uky.edu (K. Vušković)

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greedily on this order gives the upper bound:

$$\chi(G) \leqslant \beta(G),$$

where $\beta(G) = \max\{\delta_{G'} + 1\colon G'$ is an induced subgraph of $G\}$.

Markossian et al. [4] call a graph G *β-perfect* if, for each induced subgraph H of G, $\chi(H) = \beta(H)$. Thus, by definition, β-perfect graphs admit an optimal vertex-coloring algorithm. A *short-chorded* cycle has precisely one chord and this chord forms a triangle with two edges of the cycle. A *diamond* is a short-chorded cycle with four vertices. Markossian, Gasparian and Reed showed that graphs with no even hole and no short-chorded even cycle are β-perfect. They gave a recognition algorithm for the class of β-perfect graphs obtained by forbidding even holes and even cycles with exactly one chord. Markossian, Gasparian and Reed showed examples of graphs that are β-imperfect and yet contain no even hole. It would be of interest to determine the complexity of deciding if a given graph is β-perfect. This problem is known to be in co-NP.

One way of defining classes of β-perfect graphs is to consider properties of minimal β-imperfect graphs.

A vertex is *simplicial* if its neighborhood set induces a clique (note, we also consider simplicial a vertex of degree 0). A *simplicial extreme* is a vertex that is either simplicial or of degree 2. Markossian, Gasparian and Reed proved that a minimal β-imperfect graph which is not an even hole contains no simplicial extreme. Then they proved that graphs with no even hole and no short-chorded even cycle always contain a simplicial extreme.

In this paper we define a new class of β-perfect graphs (extending the results in [4]) by considering the class of graphs with no even hole, no short-chorded cycle on six vertices, and no diamond. We establish the following class inclusion:

Theorem 1.1. *If G is a graph that contains no even hole, no diamond and no short-chorded cycle on six vertices, then G is β-perfect.*

In order to establish Theorem 1.1 we prove that graphs containing no even hole, no diamond and no short-chorded cycle on six vertices always contain a simplicial extreme.

Theorem 1.2. *If G is a graph that contains no even hole, no diamond and no short-chorded cycle on six vertices, then G contains a simplicial extreme.*

In order to establish Theorem 1.2 we decompose a graph with no even hole, no diamond and no short-chorded cycle on six vertices as follows:

Theorem 1.3. *If G is a graph with no even hole, no diamond and no short-chorded cycle on six vertices, then one of the following holds:*

(i) *G is triangulated*;
(ii) *G contains a* 5*-hole and, for every edge* xy, *G has a simplicial extreme in* $G\backslash(N(x)\cup N(y))$;
(iii) *G contains no* 5*-hole and, for every edge* xy, *G has two non-adjacent simplicial extremes in* $G\backslash(N(x)\cup N(y))$.

To decide whether a graph contains an even hole was recently established to be in P by Conforti et al. [1,2]. This fact implies the existence of a polynomial-time recognition algorithm for the new class of β-perfect graphs presented here.

We conjecture that the following condition is enough to give a class of β-perfect graphs:

Conjecture 1.4. If G is a graph that contains no even hole and no diamond, then G is β-perfect.

We note that there are examples of β-imperfect graphs with no even hole and no short-chorded even cycle on at least six vertices.

2. Preliminaries

A *hole* is a chordless cycle of length greater than 3. An *n-hole* is a hole of length n. A graph is *triangulated* if it does not contain a hole.

Theorem 2.1 (Dirac [3]). *Every triangulated graph that is not a clique contains at least two non-adjacent simplicial vertices.*

The following argument shows that Theorem 1.2 implies Theorem 1.1:

Proof of Theorem 1.1. Let G be a graph that contains no even hole, no diamond and no short-chorded cycle on six vertices. It is enough to show that, for every induced subgraph G' of G, $\chi(G')\geqslant\delta_{G'}+1$. Let G' be an induced subgraph of G. By Theorem 1.2, G' contains a simplicial extreme.

First assume that G' contains a simplicial vertex x. Then $d_{G'}(x)+1\leqslant\chi(G')$ and since $\delta_{G'}+1\leqslant d_{G'}(x)+1$ we have the desired result.

Now assume that G' does not contain a simplicial vertex. Then, by Theorem 2.1, G' is not triangulated. So G' contains a hole and since by our assumption this hole cannot be even, G' contains an odd hole. This implies that $3\leqslant\chi(G')$. Since G' contains a simplicial extreme, it contains a vertex of degree 2. Hence $\delta_{G'}+1=3$ and we have the desired result. □

An alternative proof of Theorem 1.1 follows directly from Theorem 1.2 and the fact established by Markossian et al. [4]: a minimal β-imperfect graph which is not an even hole contains no simplicial extreme.

Theorem 1.2 clearly follows from Theorems 1.3 and 2.1. The rest of the paper is devoted to proving Theorem 1.3.

For graphs G and H, we say that G *contains* H, if H appears in G as an induced subgraph.

For a vertex x of G, by $N(x)$ we denote the set of neighbors of x in G. For x and y adjacent vertices of G, we say that x *sees* y, and more generally, for $M \subseteq N(x)$, we say that x *sees* M. For $S \subseteq V(G)$, we denote by $G \backslash S$ the subgraph of G induced by the node set $V(G) \backslash S$.

In a connected graph G, S is a *cutset* if $G \backslash S$ is disconnected. S is a k-*star* if it is comprised of a clique C of size k and a subset of its neighbors. We refer to C as the *center* of a k-star. We also refer to 1-star as a *star* and to a 2-star as a *double star*. If S is comprised of a clique C and all of its neighbors, it is called a *full* k-*star*.

A *wheel*, denoted by (H,x), is a graph induced by a hole H and a node $x \notin V(H)$ having at least three neighbors in H, say $x_1, \dots, x_n$. We also refer to such a wheel as an n-*wheel*. If n is even, we say that (H,x) is an *even wheel*. Node x is the *center* of the wheel. A subpath of H connecting x_i and x_j is a *sector* if it contains no intermediate node that is a neighbor of x. A *short sector* is a sector of length 1 and a *long sector* is a sector of length at least 2.

A 3-*path configuration* between distinct nodes x and y, denoted by $3PC(x,y)$, is a graph induced by three chordless paths from x to y, having no common or adjacent intermediate nodes. Note that this implies that x and y are not adjacent. When it is not necessary to specify nodes x and y, we refer to this kind of a 3-path configuration as $3PC(\cdot,\cdot)$.

A 3-*path configuration* between distinct triangles $x_1x_2x_3$ and $y_1y_2y_3$, denoted by $3PC(x_1x_2x_3, y_1y_2y_3)$, is a graph induced by three chordless paths, $P_1 = x_1, \dots, y_1$, $P_2 = x_2, \dots, y_2$ and $P_3 = x_3, \dots, y_3$, having no common nodes and such that the only adjacencies between the three paths are the edges of the two triangles. When it is not necessary to specify the nodes of the two triangles, we refer to this kind of a 3-path configuration as $3PC(\triangle, \triangle)$.

It is easy to see that graphs that do not contain an even hole cannot contain an even wheel, a $3PC(\cdot,\cdot)$ nor a $3PC(\triangle,\triangle)$. This fact will be used throughout the paper.

A *bug* is a 3-wheel with exactly two long sectors. In [1] the following decomposition theorem is shown.

Theorem 2.2 (Comforti et al. [1]). *Let G be a graph that contains a bug (H,x), with x_1, x_2 and y being the neighbors of x in H such that x_1x_2 is an edge. If G does not contain an even hole, then $N(x) \cup N(y) \backslash \{x_1\}$ (resp. $N(x) \cup N(y) \backslash \{x_2\}$) is a double star cutset separating the nodes of H.*

Lemma 2.3. *Let uv be an edge of G, $N_{uv} = N(u) \cap N(v)$, $N_u = N(u) \backslash (N_{uv} \cup \{v\})$ and $N_v = N(v) \backslash (N_{uv} \cup \{u\})$. If G is (4-hole, diamond)-free, then the nodes of N_{uv} induce a clique, the nodes of N_u (resp. N_v) induce a collection of cliques and no node of N_u (resp. N_v) sees $N_{uv} \cup N_v$ (resp. $N_{uv} \cup N_u$).*

Proof. If $x, y \in N_{uv}$ are not adjacent, then the node set $C = \{u, v, x, y\}$ induces a diamond. If $x \in N_{uv}$ sees $y \in N_u \cup N_v$, then C induces a diamond. If $x \in N_u$ sees $y \in N_v$, then C induces a 4-hole. The graph induced by the nodes of N_u cannot contain a chordless path P of length 2, since otherwise the node set $V(P) \cup \{u\}$ induces a diamond. Hence the nodes of N_u (resp. N_v) induce a collection of cliques. □

Lemma 2.4. *Let G be a graph whose nodes are partitioned into sets C and F, such that either $C = \{u\} \cup N(u)$ (i.e., C is a full star) or $C = N(u) \cup N(v)$ where uv is an edge (i.e., C is a full double star). If G is triangulated and does not contain a diamond, then a vertex of F is a simplicial vertex of G.*

Proof. We may assume w.l.o.g. that F is connected. We also may assume that F sees C, since otherwise we are done by Theorem 2.1. Let s be a node of C that sees F. Note that $s \in C \setminus \{u, v\}$. No other node of C sees F, since otherwise there is a hole or a diamond.

First assume that F is a clique. If s sees all vertices of F, then any node of F is a simplicial vertex of G. If s does not see all vertices of F, then any node of F that is not adjacent to s is a simplicial vertex of G.

Now assume that F is not a clique. Then, by Theorem 2.1 applied to F, there exist two non-adjacent nodes $a, b \in F$ that are simplicial vertices of F. Since F is connected, s cannot see both a and b, else there is a hole or a diamond. W.l.o.g. s does not see a. But then a is a simplicial vertex of G. □

Lemma 2.5. *Let G be a graph whose nodes are partitioned into sets C', C'' and F, such that C' is a maximal clique in G and C'' is a star or a double star whose center does not see F. If G is triangulated and does not contain a diamond, then a vertex of F is a simplicial extreme of G.*

Proof. We may assume w.l.o.g. that F is connected. Let G' be a subgraph of G induced by $C'' \cup F$. By Lemma 2.4 applied to G', a vertex of F, say a, is a simplicial vertex of G'. So if F does not see C', we are done. Let t be a node of C' that sees F. Since C' is a maximal clique and F is connected no other node of C' sees F (else there is a hole or a diamond). We may assume that t sees a, else we are done. We may assume that $|N_{G'}(a)| \geqslant 2$, since otherwise a is of degree at most 2 in G and we are done. The neighbors of t in G' are contained in $N_{G'}(a) \cup \{a\}$, else there is a hole or a diamond. Also, t is either adjacent to all the nodes of $N_{G'}(a) \cup \{a\}$ or the only neighbor of t in F is a. In the first case a is a simplicial vertex of G. So assume that a is the only neighbor of t in F. Note that $F \setminus \{a\} \neq \emptyset$, since $|N_{G'}(a)| \geqslant 2$ and a cannot have more than one neighbor in C''. By applying Lemma 2.4 to $G'' = G' \setminus \{a\}$, C'' and $F \setminus \{a\}$, a node of $F \setminus \{a\}$, say a', is a simplicial vertex of G''. Note that a' does not have a neighbor in C'. We may assume that a' is adjacent to a, else we are done. Since a is a simplicial vertex of G' and $|N_{G'}(a) \cup \{a\}| \geqslant 3$, a' is a simplicial vertex of G. □

3. Clean holes

Throughout the rest of the paper we assume that G is a connected graph that contains no even hole and no diamond.

Definition 3.1. A hole H is clean if the nodes of $G\backslash H$ that have a neighbor in H are of the following two types:

type W: Nodes that have exactly two neighbors in H and these two neighbors are furthermore adjacent;

type S: Nodes that have exactly one neighbor in H.

We let H denote a clean hole and we label its nodes with $1,\dots,n$. For $i=1,\dots,n$, we denote by W_i the set of type W nodes w.r.t. H that are adjacent to i and $i+1$ (indices taken modulo n), and by S_i the set of type S nodes w.r.t. H that are adjacent to i. We also define sets W and S as follows: $W=\bigcup_{i=1}^{n} W_i$ and $S=\bigcup_{i=1}^{n} S_i$.

Let F be a connected component of $G\backslash(H\cup W\cup S)$. Suppose that F sees W_i and S_j. By $P^F_{W_iS_j}$ we denote a shortest path in $F\cup W_i\cup S_j$ such that one endnode of $P^F_{W_iS_j}$ is in W_i and the other is in S_j. Note that no intermediate node of $P^F_{W_iS_j}$ is adjacent to or coincident with a node of $W_i\cup S_j$. For $i\neq j$, $P^F_{S_iS_j}$ and $P^F_{W_iW_j}$ are similarly defined. If u and v are distinct nodes of $W\cup S$ that both see F, then by P^F_{uv} we denote a shortest path from u to v in $F\cup\{u,v\}$.

Lemma 3.2. *For* $1\leqslant i,\ j\leqslant n$ *the following hold*:

(i) *the nodes of* W_i *induce a clique*;

(ii) *the nodes of* S_i *induce a collection of cliques*, *i.e.*, *the graph induced by the nodes of* S_i *does not contain a chordless path of length* 2;

(iii) *if* $u\in S_i$ *and* $v\in S_j$ *are adjacent*, *then* $i=j$;

(iv) *if* $u\in W_i$ *and* $v\in W_j$ *are adjacent*, *then* $i=j$;

(v) *No node of* W_i *is adjacent to a node of* $S_i\cup S_{i+1}$.

Proof. By Lemma 2.3, (i), (ii) and (v) hold. To prove (iii) suppose that $u\in S_i$ is adjacent to $v\in S_j$, $i\neq j$. By Lemma 2.3, i and j are not adjacent, and hence the node set $V(H)\cup\{u,v\}$ induces a 3PC(i,j). If $u\in W_i$ and $v\in W_j$ are adjacent and $i\neq j$, then the node set $V(H)\cup\{u,v\}$ either induces a 4-wheel (if $j=i+1$ or $i-1$) or a 3PC$(uii+1,vjj+1)$, hence (iv) holds. □

Lemma 3.3. *If* F *is a connected component of* $G\backslash(H\cup W\cup S)$, *then one of the following holds*:

(1) F *sees* W_i *and no node of* $(W\cup S)\backslash(W_i\cup S_i\cup S_{i+1})$;

(2) F *sees* W_i *and* S_{i+2} (*resp.* S_{i-1}), *and at most one of the sets* S_{i+1} *and* S_{i+3} (*resp.* S_i *and* S_{i-2}) *and no node of* $(W\cup S)\backslash(W_i\cup S_{i+1}\cup S_{i+2}\cup S_{i+3})$ (*resp.* $(W\cup S)\backslash(W_i\cup S_i\cup S_{i-1}\cup S_{i-2})$);

(3) *F sees W_i and S_j, $j \notin \{i-1, i, i+1, i+2\}$, and no node of $(W \cup S)\backslash(W_i \cup S_j)$;*

(4) *F sees W_i, S_j and S_{j+1}, $j, j+1 \notin \{i-1, i, i+1, i+2\}$, and no node of $(W \cup S)\backslash(W_i \cup S_j \cup S_{j+1})$;*

(5) *F sees S_i and no node of $W \cup S\backslash S_i$;*

(6) *F sees S_i and S_{i+1} and no node of $(W \cup S)\backslash(S_i \cup S_{i+1})$.*

Proof. We first observe that if F sees S_i and S_j, $i \neq j$, then $j = i+1$ or $i-1$, since otherwise the node set $V(P^F_{S_iS_j}) \cup V(H)$ induces a 3PC(i,j). So, if F does not see W, then (5) or (6) holds. Assume F sees W_i. If F also sees W_j, $i \neq j$, then the node set $V(P^F_{W_iW_j}) \cup V(H)$ induces either a 4-wheel (if $j=i+1$ or $i-1$) or a 3PC$(\triangle,\triangle)$. Hence, F does not see $W\backslash W_i$. So, if (1) does not hold, F sees S_j, $j \notin \{i, i+1\}$. If F does not see $S\backslash S_j$, then (2) or (3) holds. If F does see $S\backslash S_j$, then by the first observation (2) or (4) holds. □

For the class of graphs with no even hole and no short chorded even cycle, considered in [4], $W = \emptyset$ and if F is a connected component of $G\backslash(H \cup S)$, then it satisfies (5) or (6) of Lemma 3.3 and hence it can easily be shown that either $N(i) \cup \{i\}$ or $N(i+1) \cup \{i+1\}$ is a star cutset separating F from H.

Lemma 3.4. *If F satisfies* (1), (5) *or* (6) *of Lemma* 3.3, *then $N(i) \cup N(i+1)$ is a full double star cutset separating F from H. If F satisfies* (2) *of Lemma* 3.3, *say F sees S_{i+2}, then $N(i+1) \cup N(i+2)$ is a full double star cutset separating F from H.*

Proof. If F satisfies (1), (5) or (6) of Lemma 3.3, then trivially $N(i) \cup N(i+1)$ is a cutset. Suppose that F sees W_i and S_{i+2}. Then the node set $V(P^F_{W_iS_{i+2}}) \cup V(H)$ induces a bug with center $i+1$ and by Theorem 2.2, $N(i+1) \cup N(i+2)$ is a cutset. □

Lemma 3.5. *Let F be a connected component of $G\backslash(H \cup W \cup S)$ that satisfies* (4) *of Lemma* 3.3. *Then $C = W_i \cup \{i, i+1\} \cup S_j \cup S_{j+1} \cup W_j \cup \{j, j+1\}$ is a cutset such that $G\backslash C$ contains at least three connected components containing respectively F, $H_1 = i+2, \dots, j-1$ and $H_2 = j+2, \dots, i-1$.*

Proof. Since F satisfies (4) of Lemma 3.3, F is contained in a component of $G\backslash C$ that does not contain any node of H. We now show that H_1 and H_2 are contained in different components of $G\backslash C$. Suppose not and let $P = x_1, \dots, x_n$ be a chordless path in $G\backslash C$ such that x_1 sees H_1, x_n sees H_2 and no intermediate node of P sees $H_1 \cup H_2$.

Since H is clean, $n \geqslant 2$ and x_1 and x_n are of type W or S. By construction of P, j and $j+1$ cannot be adjacent to an intermediate node of P. Let w be a node of W_i that has a neighbor in F and let s_1 (resp. s_2) be a node of S_j (resp. S_{j+1}) that has a neighbor in F.

First we make a few observations. If x_1 is adjacent to j, then x_1 is of type W and by Lemma 3.2, x_1 is not adjacent to s_1. Similarly, if x_n is adjacent to $j+1$, then x_n is not adjacent to s_2. Also, if x_1 (resp. x_n) is of type S, then it is not adjacent to s_1 nor s_2.

Claim 1. *Nodes s_1 and s_2 do not have a neighbor in P.*

Proof. First, suppose that both s_1 and s_2 have a neighbor in P and let P' be a subpath of P such that s_1, P', s_2 is a chordless path. Then paths s_1, P', s_2; $s_1, j, j+1, s_2$ and $P^F_{s_1 s_2}$ induce a 3PC(s_1, s_2). Now suppose w.l.o.g. that s_1 has a neighbor in P and s_2 does not. Then there is a 3PC($s_1, j+1$) with two of the paths being $P^F_{s_1 s_2}, j+1$ and $s_1, j, j+1$ and the third path passing through $x_k, \ldots, x_n$ where x_k is the node of P with the largest index adjacent to s_1. This completes the proof of Claim 1. □

Claim 2. *Nodes i and $i+1$ are not adjacent to an intermediate node of P.*

Proof. W.l.o.g. suppose that $i+1$ is adjacent to an intermediate node of P. Let x_k be the node of P with highest index adjacent to $i+1$. Note that $k > 1$. If w is not adjacent to a node of $x_k, \ldots, x_n$, then there is a 3PC($i+1, j+1$) in which two of the paths are $i+1, P^F_{ws_2}, j+1$ and $i+1, H_1, j, j+1$ and the third path passes through $x_k, \ldots, x_n$. Otherwise, there is a 3PC($w, j+1$) in which two of the paths are $P^F_{ws_2}, j+1$ and $w, i+1, H_1, j, j+1$ and the third path passes through $x_l, \ldots, x_n$ (where x_l is the node of P with highest index adjacent to w). This completes the proof of Claim 2. □

Nodes x_1 and x_n cannot both be of type S, since otherwise the node set $V(P) \cup V(H)$ induces a 3PC($\cdot, \cdot$). Nodes x_1 and x_n cannot both be of type W, since otherwise $V(P) \cup V(H)$ induces a 3PC($\triangle, \triangle$). W.l.o.g. assume that x_1 is of type S and x_n of type W. Note that by Lemma 3.2, x_1 is not adjacent to s_1 nor s_2.

Claim 3. *Node w is not adjacent to a node of P.*

Proof. Suppose it is and let x_k be the node of P with highest index adjacent to w. If $k = 1$, then there is a 3PC($x_1, j+1$) with one path being $x_1, P^F_{ws_2}, j+1$ and the other two paths using the nodes of P and some of the nodes of H. Otherwise, the node set $V(H) \cup \{w, x_k, \ldots, x_n\}$ induces either a 3PC($\triangle, \triangle$) (if x_n is not adjacent to i) or a 4-wheel with center i (if x_n is adjacent to i). This completes the proof of Claim 3. □

Now, by Claims 1, 2 and 3, the node set $V(P) \cup V(P^F_{ws_2}) \cup V(H) \backslash \{k+1, \ldots, j\}$ (where k is the neighbor of x_1 in H) induces either a 3PC($\triangle, \triangle$) (if x_n is not adjacent to i) or a 4-wheel with center i (if x_n is adjacent to i). □

Lemma 3.6. *Let H be a clean hole and let P be a shortest path from W_i to S_j in $G \backslash (H \cup W \cup S \backslash (W_i \cup S_j))$. Suppose that $j \neq i, i+1$. Let H' be the hole induced by* *the node set* $V(P) \cup \{j, j+1, \ldots, i\}$. *If* H' *is not clean, then there is a bug* (H', a), *with vertex* $a \in S \cup W$, *and* a *sees* $H' \backslash P$.

Proof. Suppose H' is not clean and let a be a node of $G \backslash H'$ that has a neighbor in H' but is not of type W or S w.r.t. H'. If a has precisely two neighbors in H' then the node set $V(H') \cup \{a\}$ induces a 3PC$(\cdot,\cdot)$. Hence a sees at least three nodes of H'. Because H is clean, vertex a sees at most two vertices of H and in case a does see two vertices of H, these nodes are adjacent.

Let $P = x_1, \ldots, x_n$, where $x_1 = v$, is the endnode of P in W_i and $x_n = u$ is the endnode of P in S_j. Note that either P has length one, i.e., v is adjacent to u, or P is a $P^F_{W_i S_j}$, for some component F of $G \backslash (H \cup W \cup S)$ and no vertex of W_i sees a vertex of S_j.

Case 1: P has length one. Note that in case a sees more than four nodes of H', vertex a has to see at least three vertices of H, which contradicts H being clean. Vertex a cannot see exactly four vertices of H', since otherwise the node set $V(H') \cup \{a\}$ induces a 4-wheel with center a. So vertex a sees precisely three vertices of H', and one of those nodes is in H, which implies $a \in W \cup S$.

Note that if a belongs to S, then vertex a sees both u and v, which contradicts Lemma 3.2.

Thus vertex $a \in W$ and $a \notin W_i$, as vertex a has to see three vertices of H'. So a misses v, sees u and sees $H' \backslash P$. Now Lemma 3.2 implies $a \in W_k$, $j+1 < k < i-1$ and we have a bug (H', a).

Case 2: No vertex of W_i sees a vertex of S_j. Note that if $a \notin W \cup S$, then, because a has to see at least three vertices of H', vertex a sees F. So vertex $a \in F$ and a sees at least three vertices of P. Since we are assuming no diamond, this contradicts the minimality of P.

Thus vertex $a \in W \cup S$. Consider now the case $a \in W_k$, $k \neq i$. Lemma 3.2 says vertex a cannot see v. In addition, vertex a cannot see F, as component F cannot see W_i, W_k, with $i \neq k$. So u is the unique neighbor of vertex a in P and a has to see $H' \backslash P$. Since we are assuming no diamond, we have a bug (H', a).

Note that if $a \in W_i$, then vertex a sees both v and i. Moreover, a misses u, as we are assuming that no vertex of W_i sees a vertex of S_j. So all other vertices seen in H' by a must be in F. Since we are assuming no diamond, this contradicts the minimality of P.

It remains to consider case $a \in S$. Note that if $a \in S_j$, vertex a sees j but cannot see v, as we are assuming that no vertex of W_i sees a vertex of S_j. Now because a has to see at least three vertices of H', a has to see F. Since we are assuming no diamond, this contradicts the minimality of P.

Hence we may assume $a \in S_k$, $k \neq j$. By Lemma 3.2, vertex a does not see u. So vertex a has to see at least two nodes in P and so a sees component F. By Lemma 3.3, $k = j-1$ or $j+1$. Now suppose that a does not see two nonadjacent nodes in P. Thus a sees precisely one edge of P, a has to see $H' \backslash P$, $a \in S_{j+1}$ and we have a bug (H', a).

Otherwise, a sees two non-adjacent vertices of P. Let x_r, x_s be neighbors of a in P with lowest and highest index, respectively. If $k \neq i$, $i+1$, then there is a 3PC(a,j), induced by the node set $\{a,x_1,\dots,x_r,x_s,\dots,x_n\}$ and a subset of $V(H)$. Otherwise, $k=i$ or $i+1$, and we let H'' be the hole induced by the node set of $P \cup \{i+1,\dots,j\}$. If a has at least three neighbors in P, both (H',a) and (H'',a) are wheels, so one of the two must be an even wheel. The remaining case is that x_r and x_s are the only neighbors of a in P. Since a has at least three neighbors in H', $k=i$. But then, since $x_r x_s$ is not an edge, the node set $V(H'') \cup \{a\}$ induces a 3PC(x_r,x_s). □

Lemma 3.7. *Let H be a clean hole and F a component of $G \backslash (H \cup W \cup S)$. Suppose that there does not exist a clean hole H' such that for a component F' of $G \backslash (H' \cup W' \cup S')$, $F \subset F'$. If F satisfies* (3) *of Lemma* 3.3, *then either $C = W_i \cup S_j \cup \{i,i+1,j\}$ is a cutset such that F, $H_1 = i+2,\dots,j-1$ and $H_2 = j+1,\dots,i-1$ are contained in different components of $G \backslash C$ or there exists a full double star cutset C' such that F is contained in one of the components of $G \backslash C'$.*

Proof. Suppose that there is no full double star cutset satisfying the lemma. Since F satisfies (3) of Lemma 3.3, F is contained in a component of $G \backslash C$ that does not contain any node of H. Suppose that H_1 and H_2 are contained in the same component of $G \backslash C$. Let $P = x_1,\dots,x_n$ be a shortest path in $G \backslash C$ such that x_1 sees H_1, x_n sees H_2 and no intermediate node of P sees $H_1 \cup H_2$. Since H is clean, $n \geqslant 2$. By definition of P, j is not adjacent to an intermediate node of P. Let w (resp. s) be a node of W_i (resp. S_j) that has a neighbor in F.

Claim 1. *Node s does not have a neighbor in P. (I.e., If a node of S_j has a neighbor in F, then it does not have a neighbor in P.)*

Proof. Suppose not and consider the following two cases.

Case 1: s has a neighbor in the interior of P. If i or $i+1$ has a neighbor in the interior of P, then there exists a subpath P' of P such that one endnode of P' is adjacent to s, the other to i or $i+1$ (but not both), no intermediate node of P' has a neighbor in $\{i,i+1,s\}$ and the endnodes of P' do not have a neighbor in H. So the node set $V(H) \cup V(P') \cup \{s\}$ induces either a 3PC(i,j) (if i has a neighbor in P') or a 3PC$(i+1,j)$ (if $i+1$ has a neighbor in P'). Hence neither i nor $i+1$ has a neighbor in the interior of P.

Nodes x_1 and x_n are of different types w.r.t. H, since otherwise the node set $V(H) \cup V(P)$ induces a 3PC$(\cdot,\cdot)$ (if x_1 and x_n are of type S), a 3PC$(\triangle,\triangle)$ (if x_1 and x_n are of type W and are not both adjacent to j) or a 4-wheel with center j (if x_1 and x_n are of type W and are both adjacent to j). W.l.o.g. assume that x_1 is of type S and x_n of type W. Let a be the neighbor of x_1 in H.

If w does not have a neighbor in P, then there is a 3PC(a,s) in which two of the paths are induced by $V(P^F_{ws}) \cup V(H_1) \cup \{i+1,j\}$ and the third path uses a portion of P. If w has a neighbor in $V(P) \backslash \{x_1\}$, then the node set $V(H) \cup \{w,x_l,\dots,x_n\}$ (where x_l

is the neighbor of w in P with the highest index) induces a 3PC($\triangle,\triangle$) or a 4-wheel with center i. And finally, if the unique neighbor of w in P is x_1 then the node set $V(P^F_{ws}) \cup \{a,\dots,j,x_1,\dots,x_k\}$ (where x_k is the neighbor of s in P with lowest index, note that by Lemma 3.2, $k>1$) induces a 3PC(x_1,s).

Case 2: s does not have a neighbor in the interior of P. W.l.o.g. assume that s is adjacent to x_1. Then, by Lemma 3.2, x_1 is of type W and it is adjacent to neither j nor w. If x_1 is the unique neighbor of s in P then there is a 3PC(x_1,j) induced by $V(P)\cup\{s\}$ and a subset of $V(H)$. So s is adjacent to x_n. By Lemma 3.2, x_n is of type W and it is adjacent to neither j nor w. Suppose i has a neighbor in the interior of P. Let x_k be the neighbor of i in P with the lowest index. Note $k<n$. Then there is a 3PC(x_1,j) with two of the paths being x_1,s,j and $x_1,\dots,x_k,i,H_2,j$, and the third path passing through a portion of H_1. Hence i does not have a neighbor in the interior of P, and similarly neither does $i+1$. But then the node set $V(H)\cup V(P)$ induces a 3PC($\triangle,\triangle$). This completes the proof of Claim 1. □

Claim 2. *Nodes i and $i+1$ are not adjacent to an intermediate node of P.*

Proof. W.l.o.g. suppose that $i+1$ has a neighbor in the interior of P. Let x_k be the node of P with highest index adjacent to $i+1$. If w does not have a neighbor in $x_k,\dots,x_n$, then there is a 3PC($i+1,j$) in which two of the paths are $i+1,H_1,j$ and $i+1,P^F_{ws},j$ and the third path passes through $x_k,\dots,x_n$. If w does have a neighbor in $x_k,\dots,x_n$ then there is a 3PC(w,j) in which two of the paths are $w,i+1,H_1,j$ and P^F_{ws},j and the third path passes through $x_l,\dots,x_n$ (where x_l is the node of P with highest index adjacent to w). This completes the proof of Claim 2. □

Nodes x_1 and x_n cannot be of the same type w.r.t. H, since otherwise, by Claim 2, the node set $V(P)\cup V(H)$ induces a 3PC($\cdot,\cdot$), a 3PC($\triangle,\triangle$) or a 4-wheel with center j. W.l.o.g. assume that x_1 is of type S (with neighbor a in H) and x_n of type W (with neighbors b and $b+1$ in H).

Claim 3. *Node w does not have a neighbor in P. (I.e. If a node of W_i has a neighbor in F then it does not have a neighbor in P.)*

Proof. Suppose not and let x_k be the node of P with highest index adjacent to w. If $k=1$, then there is a 3PC(x_1,j) using nodes of P, P^F_{ws} and some nodes of H. If $k>1$, then the node set $V(H)\cup\{w,x_k,\dots,x_n\}$ induces either a 3PC($\triangle,\triangle$) (if x_n is not adjacent to i) or a 4-wheel with center i (if x_n is adjacent to i). This completes the proof of Claim 3. □

Claim 4. *Hole H' induced by the node set $V(P)\cup\{b+1,\dots,i,i+1,\dots,a\}$ is clean.*

Proof. If both x_n and a are adjacent to j, then (H',j) is a bug, so by Theorem 2.2, $C'=N(j)\cup N(a)$ is a full double star cutset such that F is contained in one of the components of $G\backslash C'$, which contradicts our assumption. Otherwise H' satisfies the hypothesis of Lemma 3.6, so there is a bug (H',z) with $z\in W\cup S$. Also z sees $H'\backslash P$, so $z\notin S_j$. If $z\in W_i$, then by Claim 3, z does not see F. If $z\notin W_i$, then since F satisfies (3) of Lemma 3.3, z does not have a neighbor in F. Let t be a common node of the two long sectors of (H',z). By Theorem 2.2, $N(z)\cup N(t)$ is a full double star cutset, which contradicts our assumption. This completes the proof of Claim 4. □

Claim 4 contradicts the assumption of the lemma because, by Claim 1, s does not have a neighbor in H', so there is a component of $G\backslash(H'\cup W'\cup S')$ that contains $F\cup\{s\}$. □

4. Proof of Theorem 1.3

The main strategy in the proof of Theorem 1.3 is to find some special cutsets, then to use induction in order to find simplicial extremes in some subgraphs related to these cutsets, and finally to show that these simplicial extremes remain simplicial extremes of the whole graph.

Proof. Assume not and let G be a minimal counterexample, i.e., suppose that the theorem holds for every proper induced subgraph of G, but it fails for G. Then G is a connected graph and is not triangulated.

We first assume that G contains a 5-hole H with vertices labeled 1, 2, 3, 4, 5. Because G has no even hole and no diamond, H is clean. In addition, because G has no short-chorded cycle on six vertices, we have no vertices of type W w.r.t. H.

In case $H\cup S$ contains all vertices of G, by Lemma 3.2, for every edge xy of G, there exists an index i, where $1\leqslant i\leqslant 5$, such that xy is of one of the following types: edge $ii+1$ (indices taken modulo 5); edge is_i, where $s_i\in S_i$; or edge s_is_i', where $s_i,s_i'\in S_i$. Now given an edge xy of G with index i, the set $\{i+3\}\cup S_{i+3}$ contains a simplicial extreme in $G\backslash(N(x)\cup N(y))$.

Otherwise, $H\cup S$ does not contain all vertices of G and we let F be a connected component of $G\backslash(H\cup S)$. We now show that F contains a simplicial extreme of G. By Lemma 3.3, for some $1\leqslant i\leqslant 5$, F sees a subset of $S_i\cup S_{i+1}$ and no node of $G\backslash(S_i\cup S_{i+1}\cup F)$. Let G' be the graph induced by the node set $\{i,i+1\}\cup S_i\cup S_{i+1}\cup F$. If G' is not triangulated, then by the minimality hypothesis on G, G' contains a simplicial extreme in $G'\backslash(N(i)\cup N(i+1))$. This simplicial extreme is in F and hence is also a simplicial extreme of G. If G' is triangulated, then by Lemma 2.4, a vertex of F is a simplicial vertex of G', and hence of G as well. So every component of $G\backslash(H\cup S)$ contains a simplicial extreme of G.

Now by Lemmas 3.2 and 3.3, for every edge xy of G, there exists an index i, where $1\leqslant i\leqslant 5$, such that either xy is of one of the following types: edge $ii+1$; edge is_i, where $s_i\in S_i$; or edge s_is_i', where $s_i,s_i'\in S_i$; or xy is of one of the following types:

edge $s_i f$, where $s_i \in S_i$ and $f \in F$; or edge ff', where $f, f' \in F$, and where F is a connected component of $G \backslash (H \cup S)$, such that F sees a subset of $S_i \cup S_{i+1}$ and no node of $G \backslash (S_i \cup S_{i+1} \cup F)$.

In any case, if there is no connected component F' of $G \backslash (H \cup S)$ that sees S_{i+3}, then by Lemma 3.2, a node of $\{i+3\} \cup S_{i+3}$ is a simplicial extreme of G that is in $G \backslash (N(x) \cup N(y))$, and otherwise a node of F' is a simplicial extreme of G that is in $G \backslash (N(x) \cup N(y))$.

So we may assume now that G contains no 5-hole.

Throughout the proof of this theorem, when we say that C is a cutset of G, we denote by $C_1, \ldots, C_m$ the connected components of $G \backslash C$ and for $i = 1, \ldots, m$ we denote by G_i the subgraph of G induced by $C \cup C_i$.

Claim 1. *G does not contain a clique cutset.* (*In particular*, *G does not contain a simplicial vertex.*)

Proof. Suppose C is a clique cutset of G. Let xy be an edge of G. We now show that

(1) there exist two nonadjacent simplicial extremes of G that are in $G \backslash (N(x) \cup N(y))$.

Case 1: $xy \in G \backslash C$. W.l.o.g. assume that $xy \in C_1$. First suppose that one of the graphs $G_2, \ldots, G_m$ is not triangulated, say G_2 is not. Since the theorem holds for G_2, there exist two nonadjacent simplicial extremes of G_2, say a and b, that are in $G_2 \backslash (N(u) \cup N(v))$ (where u is a node of C that has a neighbor v in C_2). Since C is a clique, a and b are in C_2, and hence are simplicial extremes of G as well. Since nodes of C_1 do not see C_2, (1) holds.

Now assume that $G_2, \ldots, G_m$ are all triangulated. Then G_1 is not, since otherwise G would be triangulated. Since the theorem holds for G_1, there exist two nonadjacent simplicial extremes of G_1 in $G_1 \backslash (N(x) \cup N(y))$. Since C is a clique, one of these simplicial extremes, say a, is in C_1 and hence is a simplicial extreme of G. Since G_2 is triangulated and C is a clique, by Theorem 2.1, there exists a simplicial vertex b of G_2 in C_2. Since b is in C_2, it is also a simplicial vertex of G. So, a and b are nonadjacent simplicial extremes of G that are in $G \backslash (N(x) \cup N(y))$, hence (1) holds.

Case 2: $x \in C$, $y \notin C$. W.l.o.g. assume that $y \in C_1$. First suppose that one of the graphs $G_2, \ldots, G_m$ is not triangulated, say G_2 is not. Let u be a node of C that has a neighbor v in C_2 (if x has a neighbor in C_2, let $u = x$). Since the theorem holds for G_2, there exist two nonadjacent simplicial extremes of G_2, say a and b, that are in $G_2 \backslash (N(u) \cup N(v))$. Since C is a clique, $a, b \in C_2$ and hence they are simplicial extremes of G, so (1) holds.

Now assume that all of the graphs $G_2, \ldots, G_m$ are triangulated. Then G_1 is not. Since G_1 satisfies the theorem, there exist two nonadjacent simplicial extremes of G_1, say a and b, that are in $G_1 \backslash (N(x) \cup N(y))$. Since C is a clique, $a, b \in C_1$, so they are also simplicial extremes of G and hence (1) holds.

Case 3: $x, y \in C$. Since G is not triangulated, one of the graphs $G_1, \dots, G_m$ is not triangulated, say G_1 is not. So there exist two nonadjacent simplicial extremes of G_1, say a and b, that are in $G_1 \backslash (N(x) \cup N(y))$. Since C is a clique, a and b are in C_1 and hence are simplicial extremes of G, so (1) holds.

This completes the proof of Claim 1. □

Let xy be an edge of G. We obtain a contradiction by showing that

(∗) *there exist two nonadjacent simplicial extremes of* G *that are in* $G \backslash (N(x) \cup N(y))$.

Claim 2. *Let* (H', u) *be a bug, with* u_1, u_2 *and* v *being the neighbors of* u *in* H' *such that* $u_1 u_2$ *is an edge. If one of the following holds*:

(i) $x = u$ *and* $y = v$,
(ii) $x = u$ *and* $y = u_2$,
(iii) $x \notin N(u) \cup N(v)$ *and* $y = u_2$, *or*
(iv) $x, y \notin N(u) \cup N(v)$,
then (∗) *holds.*

Proof. By Theorem 2.2, $C = N(u) \cup N(v)$ is a cutset such that if C_1 and C_2 are components of $G \backslash C$ that contain nodes of H', then w.l.o.g. u_2 has a neighbor in C_2 and it does not have a neighbor in C_1. Since the theorem holds for G_1 and G_1 is not triangulated, there exist two nonadjacent simplicial extremes of G_1, say a and b, that are in $G_1 \backslash C$. Since a and b are in C_1, they are simplicial extremes of G. If (i), (ii) or (iii) hold, then $a, b \notin N(x) \cup N(y)$, and so (∗) holds. If (iv) holds, then w.l.o.g. we can assume that $x, y \notin C_1$ and hence $a, b \notin N(x) \cup N(y)$, so (∗) holds. This completes the proof of Claim 2. □

Claim 3. *If* $C = N(u) \cup N(v)$ *is a full double star cutset of* G *such that* $xy \in G \backslash C$, *then* (∗) *holds.*

Proof. W.l.o.g. assume that $xy \in C_1$. First assume that G_2 is triangulated. By Lemma 2.4 applied to G_2 and C, there is a simplicial vertex of G_2, say a, that is in C_2. Since $a \in C_2$, a is also a simplicial vertex of G and hence Claim 1 is contradicted. So G_2 is not triangulated. But then, since G_2 satisfies the theorem, there exist two nonadjacent simplicial extremes of G_2, say a and b, that are in $G_2 \backslash (N(u) \cup N(v))$. Nodes a and b must be in C_2 and so are simplicial extremes of G, hence (∗) holds. This completes the proof of Claim 3. □

Claim 4. *If* C *is a cutset of* G *such that* C *is comprised of twonode disjoint sets* C' *and* C'', *where* C' *is a maximal clique of* G *and* C'' *is a star or a double star whose center does not see any node of* $G \backslash C$, *then every component of* $G \backslash C$ *contains a node that is a simplicial extreme of* G.

Proof. By symmetry it is enough to show that C_1 contains a node that is a simplicial extreme of G_1 (note that such a node is also a simplicial extreme of G). If G_1 is triangulated, the result follows from Lemma 2.5. So assume that G_1 is not triangulated. If C'' is a double star let uv be the center of C'' and if C'' is a star let u be the center of C'' and v a neighbor of u. Since the theorem holds for G_1, there exist two nonadjacent simplicial extremes of G_1, say a and b, that are in $G_1\backslash(N(u)\cup N(v))$. Nodes a and b are in $C'\cup C_1$, and since C' is a clique, at least one of them is in C_1. This completes the proof of Claim 4. □

Claim 5. *If H is a clean hole of G and F is a component of $G\backslash(H\cup W\cup S)$, then a node of F is a simplicial extreme of G.*

Proof. First suppose that F satisfies (1), (5) or (6) of Lemma 3.3. Let G' be a subgraph of G induced by the node set $F\cup W_i\cup S_i\cup S_{i+1}\cup\{i,i+1\}$. G' is not triangulated since otherwise, by Lemma 2.4, a node of F is a simplicial vertex of G' (and hence of G as well), so Claim 1 is contradicted. Since G' satisfies the theorem, there exists a simplicial extreme of G' that is in $G'\backslash(N(i)\cup N(i+1))$, i.e., it is in F and hence is also a simplicial extreme of G.

Now assume that F satisfies (2), (3) or (4) of Lemma 3.3. If F satisfies (2), then let $C=W_i\cup S_{i+1}\cup S_{i+2}\cup S_{i+3}\cup\{i,i+1,i+2,i+3\}$ and otherwise let $C=W_i\cup S_j\cup S_{j+1}\cup\{i,j,j+1\}$. The node set C is a cutset of G such that F is one of the connected components of $G\backslash C$, so the result follows from Claim 4. This completes the proof of Claim 5. □

Claim 6. *If H is a clean hole, then the following hold*:

(i) *For every $u\in S_i$, u sees $G\backslash(H\cup W\cup S)$ or $W\backslash(W_{i-1}\cup W_i)$.*
(ii) *For every $u\in W_i$, u sees $G\backslash(H\cup W\cup S)$ or $S\backslash(S_i\cup S_{i+1})$.*

Proof. To prove (i), let $u\in S_i$ and suppose that u does not see $G\backslash(H\cup W\cup S)$. If u does not see $(W\cup S)\backslash S_i$, then by Lemma 3.2, u is a simplicial vertex, which contradicts Claim 1. Hence u sees $(W\cup S)\backslash S_i$. By Lemma 3.2, u cannot see $(S\backslash S_i)\cup(W_{i-1}\cup W_i)$, so u sees $W\backslash(W_{i-1}\cup W_i)$. To prove (ii), let $u\in W_i$ and suppose that u does not see $G\backslash(H\cup W\cup S)$. If u does not see $(W\cup S)\backslash W_i$, then by Lemma 3.2, u is a simplicial vertex, which contradicts Claim 1. Hence u sees $(W\cup S)\backslash W_i$. By Lemma 3.2, u cannot see $(W\backslash W_i)\cup(S_i\cup S_{i+1})$, so u sees $S\backslash(S_i\cup S_{i+1})$. This completes the proof of Claim 6. □

Claim 7. *If there exists a clean hole H so that xy is contained in a component F of $G\backslash(H\cup W\cup S)$, then $(*)$ holds.*

Proof. We may assume that H and F are picked so that there does not exist a clean hole H' such that for a component F' of $G\backslash(H'\cup W'\cup S')$, $F\subset F'$. We also assume that the hypothesis of Claim 3 does not hold, since otherwise we are done.

Case 1: F satisfies (1), (2), (5) or (6) of Lemma 3.3. Then by Lemma 3.4, there exists a full double star cutset C such that F is contained in $G \backslash C$, which contradicts our assumption.

Case 2: F satisfies (3) of Lemma 3.3. Let $P_1 = i+1, \ldots, j$ and $P_2 = j, \ldots, i$. Since G does not contain a 5-hole, not both P_1 and P_2 can be of length 2. So $C = W_i \cup N(j) \cup \{i, i+1, j\}$ is a cutset such that F is contained in $G \backslash C$, w.l.o.g. $F = C_1$. If $G \backslash C$ contains at least three connected components, then by Claim 4, C_2 and C_3 both contain a simplicial extreme of G, so $(*)$ holds. So assume that $G \backslash C$ contains only two components.

P_1 and P_2 cannot both be of length greater than 2, since otherwise, by Lemma 3.7, $G \backslash C$ contains at least three components. W.l.o.g. assume that P_1 is of length 2. By Claim 4, some $a \in C_2$ is a simplicial extreme of G. If $i+2$ is of degree 2, then $i+2$ and a are two nonadjacent simplicial extremes of G, so $(*)$ holds. So assume that $i+2$ is not of degree 2. Then $W_{i+1} \cup S_{i+2} \cup W_{i+2} \neq \emptyset$. But $W_{i+1} \cup S_{i+2} = \emptyset$, since otherwise, by Lemma 3.7, $G \backslash C$ contains at least three components. Hence $W_{i+2} \neq \emptyset$. Let $u \in W_{i+2}$. Node u cannot have a neighbor in $G \backslash (H \cup S_i \cup S_{i+1} \cup S_{i+3} \cup W_i \cup W_{i+2} \cup W_{i+3})$, since otherwise, by Lemma 3.7, $G \backslash C$ contains at least three components. By Claim 6, u has a neighbor $v \in S_i \cup S_{i+1}$. But then, if $v \in S_i$ then $u, v, i, i+1, i+2, u$ is a 5-hole and otherwise $u, v, i+1, i+2, u$ is a 4-hole.

Case 3: F satisfies (4) of Lemma 3.3. Let $P_1 = i+1, \ldots, j$ and $P_2 = j+1, \ldots, i$. P_1 and P_2 cannot both be of length 2, since otherwise H is a 6-hole. So $C = W_i \cup \{i, i+1\} \cup N(j) \cup N(j+1)$ is a cutset such that F is contained in $G \backslash C$, say $F = C_1$. If $G \backslash C$ contains at least three components, then by Claim 4, C_2 and C_3 both contain a simplicial extreme of G, so $(*)$ holds. So assume that $G \backslash C$ contains only two components. P_1 and P_2 cannot both be of length greater than 2, since otherwise, by Lemma 3.5, $G \backslash C$ contains at least three components. W.l.o.g. assume that P_1 is of length 2. By Claim 4, some $a \in C_2$ is a simplicial extreme of G. If $i+2$ is of degree 2, then $i+2$ and a are two nonadjacent simplicial extremes of G, so $(*)$ holds. So assume that $i+2$ is not of degree 2. Then $W_{i+1} \cup S_{i+2} \cup W_{i+2} \neq \emptyset$. But $W_{i+1} \cup S_{i+2} = \emptyset$, since otherwise, by Lemma 3.5, $G \backslash C$ contains at least three components. Hence $W_{i+2} \neq \emptyset$. Let $u \in W_{i+2}$. Node u cannot have a neighbor in $G \backslash (H \cup S_i \cup S_{i+1} \cup S_{i+3} \cup S_{i+4} \cup W_i \cup W_{i+2} \cup W_{i+3})$, since otherwise, by Lemma 3.5, $G \backslash C$ contains at least three components. By Claim 6, u has a neighbor in $S_i \cup S_{i+1} \cup S_{i+4}$. But then there is a 4- or 5-hole.

This completes the proof of Claim 7. □

Claim 8. *If there exists a clean hole H that contains both x and y, then $(*)$ holds.*

Proof. We may assume that the hypotheses of Claims 2 and 7 do not hold, else we are done. W.l.o.g. assume that $x = 1$ and $y = 2$. □

Claim 8.0. *There does not exist a hole H' which contains no node of $N(x) \cup N(y)$ and such that H' consists of a shortest path from W_i to S_j, with $j \neq i$, $i+1$, in $G \backslash ((H \cup W \cup S) \backslash (W_i \cup S_j))$ and the node set $\{j, j+1, \ldots, i\}$.*

Proof. Suppose that such a hole H' exists. H' is not clean, since otherwise our assumption is contradicted since $xy \in G \backslash (H' \cup W' \cup S')$. By Lemma 3.6, there is a bug (H', u), which again contradicts our assumption. This completes the proof of Claim 8.0. □

Claim 8.1. *One of the following holds*:

(i) *vertex* 4 *is of degree* 2,
(ii) $W_3 \cup S_4 \cup W_4$ *sees* $G \backslash (H \cup W \cup S)$, *or*
(iii) W_4 *sees* S_n.

Proof. Suppose that (i) and (ii) do not hold. Then there exists a node u in $W_3 \cup S_4 \cup W_4$. First assume that $u \in S_4$. By Claim 6, u sees $v \in W_i$, $i \neq 3, 4$. If $i = 2, 1$ or n, then there is a 4-, 5- or 6-hole. Otherwise, the hole $H' = u, 4, \ldots, i, v, u$ contradicts Claim 8.0. Now assume that $u \in W_3$. By Claim 6, u sees $v \in S_i$, $i \neq 3, 4$. If $i = 2, 1$ or n, then there is a 4-, 5- or 6-hole. Otherwise, the hole $H' = u, 4, \ldots, i, v, u$ contradicts Claim 8.0. Finally assume that $u \in W_4$. By Claim 6, u sees $v \in S_i$, $i \neq 4, 5$. Suppose that $i \neq n$. If $i = 3, 2$ or 1, then there is a 4-, 5- or 6-hole. Otherwise, the hole $H' = u, 5, \ldots, i, v, u$ contradicts Claim 8.0. This completes the proof of Claim 8.1. □

Claim 8.2. *If* $u \in W_4$ *sees* $v \in S_n$, *then one of the following holds*:

(i) v *is of degree* 2,
(ii) v *sees* $G \backslash (H \cup W \cup S)$, *or*
(iii) v *does not see* $G \backslash (H \cup W \cup S)$, *but a neighbor* v' *of* v *in* S_n *sees* $G \backslash (H \cup W \cup S)$.

Proof. Suppose that (i) and (ii) do not hold. Let $H' = u, 5, \ldots, n, v, u$. First assume that v does not have a neighbor in S_n. Then by Claim 6, v sees $w \in W \backslash (W_{n-1} \cup W_n \cup \{u\})$. Node w is not in $W_1 \cup W_2 \cup W_3$, since otherwise there is a 4-, 5- or 6-hole. But then (H', w) is a bug (since G does not contain a 4-hole and G does not contain a diamond), which contradicts our assumption. Now assume that v has a neighbor $v' \in S_n$. Suppose v' does not see $G \backslash (H \cup W \cup S)$. By Claim 6, v' has a neighbor $w \in W_i$, $i \neq n-1, n$. As above $i \neq 1, 2, 3$. Also v does not see w, since otherwise the node set $\{n, v, v', w\}$ induces a diamond. Hence the node set $V(H') \cup \{v', w\}$ induces a 3PC($\triangle, \triangle$). This completes the proof of Claim 8.2. □

By Claim 8.1 and its symmetric case, we need to consider only the following three cases.

Case 1: $n-1$ is of degree 2. If 4 is also of degree 2, then $(*)$ holds. If $W_3 \cup S_4 \cup W_4$ sees a component F of $G \backslash (H \cup W \cup S)$, then by Claim 5, F contains a simplicial extreme, so $(*)$ holds. Otherwise, by Claim 8.1, $u \in W_4$ sees $v \in S_n$. If v is of degree 2, then $(*)$ holds. Otherwise, by Claim 8.2, S_n sees $G \backslash (H \cup W \cup S)$, so $(*)$ holds by Claim 5.

Case 2: $u \in W_{n-2}$ sees $v \in S_3$. By symmetry applied to Claim 8.2, either v is of degree 2, or v or a neighbor of v in S_3 sees $G \backslash (H \cup W \cup S)$. First assume that v is of degree 2. If 4 is also of degree 2, then $(*)$ holds. If $W_3 \cup S_4 \cup W_4$ sees $G \backslash (H \cup W \cup S)$, then $(*)$ holds by Claim 5. Otherwise, by Claim 8.1, $u' \in W_4$ sees $v' \in S_n$. But then, by Lemma 3.2 and since G cannot contain a 4-hole, the node set $(V(H) \backslash \{x, y\}) \cup \{u, v, u', v'\}$ induces either a 3PC$(\triangle, \triangle)$ (if $n-2 \neq 5$) or a 4-wheel with center 5. Hence we can assume that v is not of degree 2.

If v sees $G \backslash (H \cup W \cup S)$, let F be a component of $G \backslash (H \cup W \cup S)$ that v sees. Otherwise let v'' be a neighbor of v in S_3 and F a component of $G \backslash (H \cup W \cup S)$ that v'' sees (and v does not). If 4 is of degree 2, then by Claim 5 applied to F, $(*)$ holds. As above $u' \in W_4$ cannot see $v' \in S_n$, so by Claim 8.1, we can assume that $W_3 \cup S_4 \cup W_4$ sees a component F' of $G \backslash (H \cup W \cup S)$. If $F \neq F'$, then by Claim 5 applied to F and F', $(*)$ holds. So assume $F = F'$ and consider the following two cases.

Case 2.1: S_4 sees F. W_{n-2} does not see F, since otherwise the hole induced by the node set $V(P^F_{S_4 W_{n-2}}) \cup \{4, \ldots, n-2\}$ contradicts Claim 8.0. Let v' be a node of S_4 that sees F. Node u does not see v', since otherwise $v', 4, 3, 2, 1, n, n-1, u, v'$ is an 8-hole. If v sees F, then the node set $V(P^F_{vv'}) \cup \{u, v, v', 3, \ldots, n-2\}$ induces a 3PC$(4, v)$. So v does not see F and v'' does.

If F does not see W_i, for all $i \neq 3, 4$, then let G' be the subgraph of G induced by the node set $S_3 \cup S_4 \cup W_3 \cup W_4 \cup F \cup \{3, 4\}$. Since G is a minimal counterexample and G' is not triangulated, there exist two nonadjacent simplicial extremes of G' in $G' \backslash (N(3) \cup N(4))$. These simplicial extremes must be in F and so are simplicial extremes of G as well by Lemma 3.3, hence $(*)$ holds.

Now suppose that F sees W_i, for an $i \neq 3, 4$. If $4 < i \leqslant n-1$, then there exists a hole that contradicts Claim 8.0. Let w be a node of W_i that sees F. If $i = 1$ or n, then v does not see w (else there is a 4- or a 5-hole) and the node set $S = V(P^F_{wv''}) \cup \{u, v, n-1, n, 1, 2, 3\}$ induces a 3PC$(\triangle, \triangle)$. If $i = 2$, then S induces a 4-wheel with center 3.

Case 2.2: S_4 does not see F, and $W_3 \cup W_4$ does. If $w \in W_4$ sees F, then the node set $V(P^F_{v''w}) \cup \{v, u, 3, 4, \ldots, n-2\}$ induces a 3PC$(\triangle, \triangle)$. Hence W_4 does not see F, and W_3 does. By Lemma 3.3, no node of $(W \cup S) \backslash (S_2 \cup S_3 \cup W_3)$ sees F. Let G' be the subgraph of G induced by the node set $F \cup S_2 \cup S_3 \cup W_3 \cup \{2, 3, 4\}$. Since G is a minimal counterexample and G' is not triangulated, there are two nonadjacent simplicial extremes of G' in $G' \backslash (N(2) \cup N(3))$. These simplicial extremes must be in F and so are simplicial extremes of G as well by Lemma 3.3, hence $(*)$ holds.

Case 3: There exists a component F (resp. F') of $G \backslash (H \cup W \cup S)$ that sees $W_3 \cup S_4 \cup W_4$ (resp. $W_{n-1} \cup S_{n-1} \cup W_{n-2}$). If $F \neq F'$, then by Claim 5 applied to F and F', $(*)$ holds. So assume $F = F'$. By Lemma 3.3, F must see W. W.l.o.g. assume that F sees W_3. Then F sees S_{n-1} and the hole H' induced by the node set $V(P^F_{W_3 S_{n-1}}) \cup \{4, \ldots, n-1\}$ contradicts Claim 8.0.

This completes the proof of Claim 8. $\square$

Claim 9. *If there exists a clean hole H such that $x \in W$ and $y \in G \backslash (H \cup W \cup S)$, then $(*)$ holds.*

Proof. We may assume that the hypotheses of Claims 2, 7 and 8 do not hold, since otherwise we are done. W.l.o.g. assume that x is adjacent to 1 and 2. □

Claim 9.0. *If $u \in W_2 \cup S_3 \cup W_3$ (resp. $u \in W_{n-1} \cup S_n \cup W_n$), then u does not see x nor y.*

Proof. If $u \in W_2 \cup W_3$, then by Lemma 3.2, u does not see x and by Lemma 3.3, u does not see y. If $u \in S_3$ sees x, then $u, x, 2, 3, u$ is a 4-hole, and if it sees y, then $u, y, x, 2, 3, u$ is a 5-hole. This completes the proof of Claim 9.0. □

Claim 9.1. *Either 3 (resp. n) is of degree 2, or there is a component F of $G \backslash (H \cup W \cup S)$ that sees $W_2 \cup S_3 \cup W_3$ (resp. $W_{n-1} \cup S_n \cup W_n$) and is such that x and y do not see F.*

Proof. Suppose that 3 is not of degree 2. Then there is a node $u \in W_2 \cup S_3 \cup W_3$. By Claim 9.0, u does not see x nor y.

Case 1: $u \in W_2$. If there is a component F of $G \backslash (H \cup W \cup S)$ that sees W_2, then by Lemma 3.3, F does not see W_1, and hence F does not see x nor y, so we are done. Otherwise, by Claim 6, u sees $v \in S_i$, $i \neq 2, 3$. Node v does not see x, else $u, v, x, 2, u$ is a 4-hole. Node v does not see y, else $u, v, y, x, 2, u$ is a 5-hole. If $i = 1$, then there is a 4-hole. Hence the hole $H' = u, 3, \ldots, i, v, u$ is such that nodes x and y do not see H'. H' is not clean, else our assumption is contradicted. By Lemma 3.6, there is a bug (H', a), with $a \in W_k$ or $a \in S_k$ for some $3 < k < i$. If a sees neither x nor y, then (H', a) contradicts our assumption. If x sees a, then by Lemma 3.2, $a \in S_k$ and hence is adjacent to both u and v. But then $x, a, u, 2, x$ is a 4-hole. If y sees a, then $a \in S_k$, since otherwise the node set $V(H) \cup \{x, y, a\}$ induces a 3PC($\triangle, \triangle$). So a sees both u and v, and hence $x, 2, u, a, y, x$ is a 5-hole.

Case 2: $u \in W_3$. As in Case 1, we may assume that u sees $v \in S_i$, $i \neq 3, 4$. If $i = 2$ or 1, then there is a 4- or 5-hole. If v sees x, then $u, v, x, 2, 3, u$ is a 5-hole. If v sees y, then $u, v, y, x, 2, 3, u$ is a 6-hole. Hence the hole $H' = u, 4, \ldots, i, v, u$ does not see x nor y. H' is not clean, else our assumption is contradicted. By Lemma 3.6, there is a bug (H', a) with $a \in W_k$ or $a \in S_k$ for some $4 < k < i$. If a sees neither x nor y, then (H', a) contradicts our assumption. If x sees a, then by Lemma 3.2, $a \in S_k$ and hence is adjacent to both u and v. But then $x, 2, 3, u, a, x$ is a 5-hole. If a sees y, then $a \in S_k$, since otherwise the node set $V(H) \cup \{x, y, a\}$ induces a 3PC($\triangle, \triangle$). So a sees both u and v, and hence $x, 2, 3, u, a, y, x$ is a 6-hole.

Case 3: $u \in S_3$. By Claim 6, u sees $G \backslash (H \cup W \cup S)$ or $W \backslash (W_2 \cup W_3)$. First suppose that u sees a component F of $G \backslash (H \cup W \cup S)$. We may assume that x sees F, else we are done. Then the node set $V(P^F_{xu}) \cup V(H)$ induces a bug with center 2, which contradicts our assumption. Now suppose that u sees $v \in W_i$, $i \neq 2, 3$. If $i = 1$, then there is a 4-hole. By Lemma 3.2, v does not see x and by Lemma 3.3, v does not see y. Hence the hole $H' = u, 3, \ldots, i, v, u$ does not see x nor y. H' is not clean, else our assumption is contradicted. By Lemma 3.6, there is a bug (H', a) with $a \in W_k$ or $a \in S_k$ for some $3 < k < i$. If a sees nether x nor y, then (H', a) contradicts our assumption. If x sees a, then by Lemma 3.2, $a \in S_k$. So a sees both u and v, and hence $x, 2, 3, u, a, x$ is a 5-hole. If y sees a, then $a \in S_k$, else the node set $V(H) \cup \{x, y, a\}$ induces a 3PC($\triangle, \triangle$). So a sees both u and v, and hence $x, 2, 3, u, a, y, x$ is a 6-hole.

This completes the proof of Claim 9.1. □

If 3 and n are both of degree 2, then (∗) holds. W.l.o.g. assume that 3 is not of degree 2. By Claim 9.1, there is a component F of $G \backslash (H \cup W \cup S)$ that sees $W_2 \cup S_3 \cup W_3$ and does not see x nor y. By Claim 5, F contains a simplicial extreme of G. If n is of degree 2, then (∗) holds. Else, by Claim 9.1, there is a component F' of $G \backslash (H \cup W \cup S)$ that sees $W_{n-1} \cup S_n \cup W_n$ and does not see x nor y. If $F \neq F'$, then (∗) holds by Claim 5. So assume that $F = F'$. By Lemma 3.3, w.l.o.g. F sees S_n and $W_2 \cup W_3$. Again w.l.o.g. we assume that F sees W_2. Let H' be the hole induced by the node set $V(P^F_{W_2 S_n}) \cup \{3, \ldots, n\}$. Let v (resp. u) be a node of W_2 (resp. S_n) that is in H'. By Claim 9.0, v and u do not see x and y. Hence H' does not see x nor y. H' is not clean, else our assumption is contradicted. By Lemma 3.6, there is a bug (H', a) with $a \in W_k$ or $a \in S_k$ for some $3 \leqslant k < n$. If a sees neither x nor y, then (H', a) contradicts our assumption. If x sees a, then by Lemma 3.2, $a \in S_k$ and hence the node set $(V(H') \backslash \{3, \ldots, n-1\}) \cup \{a, x, 1, 2\}$ induces a 3PC($\triangle, \triangle$). If y sees a, then $a \in S_k$, since otherwise the node set $V(H) \cup \{a, x, y\}$ induces a 3PC($\triangle, \triangle$). But then the node set $(V(H') \backslash \{3, \ldots, n-1\}) \cup \{a, x, y, 1, 2\}$ induces a 3PC($\triangle, \triangle$). This completes the proof of Claim 9. □

Claim 10. *If there exists a clean hole H such that $x \in H$ and $y \in W$, then* (∗) holds.

Proof. We may assume that the hypotheses of Claims 2, 7–9 do not hold, else we are done. W.l.o.g. assume that $x = 1$ and $y \in W_1$.

Claim 10.1. *y does not see $(W \cup S) \backslash W_1$.*

Proof. Suppose not. Then by Lemma 3.2, y sees $v \in S_i$, $i \neq 1, 2$. Let $H' = v, i, \ldots, n, x, y, v$. H' is not clean, else our assumption is contradicted. By Lemma 3.6, there is a bug (H', a). If a is adjacent to both v and y, then (H', a) is a bug that contradicts our assumption. If a is adjacent to y but not to v, then $a \in W \backslash W_1$, which contradicts Lemma 3.2. So a is adjacent to v but not to y. Then $a \in W_j$, $i < j \leqslant n$. Let $H'' = a, j+1, \ldots, x, \ldots, i, v, a$. But now (H'', y) is a bug contradicting our assumption. This completes the proof of Claim 10.1. □

Claim 10.2. *If $u \in W_i$ sees $v \in S_j$, then the hole $H' = u, i+1, \ldots, j, v, u$ must contain a node of $N(x) \cup \{x\}$.*

Proof. Suppose that H' does not contain a node of $N(x) \cup \{x\}$. Suppose that H' is not clean. Then by Lemma 3.6, there is a bug (H', a). Hence $a \in W \cup S$. By Claim 10.1, y does not see $\{u, v, a\}$ and so (H', a) is a bug that contradicts our assumption. Hence H' is clean, but since $x, y \in G \backslash (H' \cup W' \cup S')$ this contradicts our assumption. This completes the proof of Claim 10.2. □

Claim 10.3. *One of the following holds*:

(i) *three is of degree* 2,
(ii) *there is a component of* $G \backslash (H \cup W \cup S)$ *that sees* $W_2 \cup S_3 \cup W_3$ *and does not see* y, *or*
(iii) *there is a component of* $G \backslash (H \cup W \cup S)$ *that sees* S_3 and y.

Proof. Suppose that (i) does not hold. Then there is a node $u \in W_2 \cup S_3 \cup W_3$.

Case 1: $u \in W_2$. If u sees a component F of $G \backslash (H \cup W \cup S)$, then by Lemma 3.3, y does not see F and hence (ii) holds. Otherwise, by Claim 6, u sees $v \in S_i$, $i \neq 2, 3$. If $i = 1$, then $u, v, 1, 2, u$ is a 4-hole. If $i = n$, then $u, v, n, 1, 2, u$ is a 5-hole. Hence $i \neq 1, 2, 3, n$ and the hole $H' = u, 3, \ldots, i, v, u$ contradicts Claim 10.2.

Case 2: $u \in W_3$. As in Case 1, we may assume that u sees $v \in S_i$, $i \neq 3, 4$. If $i = 2$, 1 or n, then there is a 4-, 5- or 6-hole. Hence $i \neq 1, 2, 3, 4, n$ and the hole $H' = u, 4, \ldots, i, v, u$ contradicts Claim 10.2.

Case 3: $u \in S_3$. If u sees a component of $G \backslash (H \cup W \cup S)$, then (ii) or (iii) holds. Otherwise, by Claim 6, u sees $v \in W_i$, $i \neq 2, 3$. If $i = 1$ or n, then there is a 4-hole or 5-hole. Hence $i \neq 1, 2, 3, n$ and the hole $H' = u, 3, \ldots, i, v, u$ contradicts Claim 10.2.

This completes the proof of Claim 10.3 □

Claim 10.4. *One of the following holds*:

(i) $n - 1$ *is of degree* 2,
(ii) *there is a component of* $G \backslash (H \cup W \cup S)$ *that sees* $W_{n-2} \cup S_{n-1} \cup W_{n-1}$ *and does not see* y, *or*
(iii) *there is a component of* $G \backslash (H \cup W \cup S)$ *that sees* S_{n-1} *and* y.

Proof. Suppose that (i) does not hold. Then there is a node $u \in W_{n-2} \cup S_{n-1} \cup W_{n-1}$.

Case 1: $u \in W_{n-2}$. If u sees a component F of $G \backslash (H \cup W \cup S)$, then by Lemma 3.3, y does not see F and hence (ii) holds. Otherwise, by Claim 6, u sees $v \in S_i$, $i \neq n-1, n-2$. If $i = n$, 1 or 2, then there is a 4-, 5- or 6-hole. Hence $i \neq 1, 2, n, n-1, n-2$ and the hole $H' = u, v, i, \ldots, n-2, u$ contradicts Claim 10.2.

Case 2: $u \in W_{n-1}$. As in Case 1, we may assume that u sees $v \in S_i$, $i \neq n-1, n$. If $i = 1$ or 2, then there is a 4- or 5-hole. Hence $i \neq 1, 2, n-1, n$ and the hole $H' = u, v, i, \ldots, n-1, u$ contradicts Claim 10.2.

Case 3: $u \in S_{n-1}$. If u sees a component of $G \backslash (H \cup W \cup S)$, then (ii) or (iii) holds. Otherwise, by Claim 6, u sees $v \in W_i$, $i \neq n-2, n-1$. If $i = n$ or 1, then there is a 4- or 5-hole. Hence $i \neq 1, n-2, n-1, n$ and the hole $H' = u, v, i+1, \ldots, n-1, u$ contradicts Claim 10.2.

This completes the proof of Claim 10.4. □

Claim 10.5. *If there exists a component F of $G \backslash (H \cup W \cup S)$ that sees y and S_i, $i \neq 1,2$, then there does not exist a component F' of $G \backslash (H \cup W \cup S)$ that sees y and S_j, $j \neq i, 1, 2$.*

Proof. Firstly, we observe that if a component F of $G \backslash (H \cup W \cup S)$ sees y and $s \in S_i$, $i \neq 1,2$, then $i \neq n$, since otherwise the node set $V(P^F_{ys}) \cup V(H)$ induces a bug with center x, which contradicts our assumption.

Secondly, we show that there does not exist a component F of $G \backslash (H \cup W \cup S)$ that sees y, S_i and S_j, $i,j \neq 1,2$ and $i \neq j$. Assume otherwise and let F be such a component. By the first observation $i,j \neq n$. By Lemma 3.3, w.l.o.g. $j = i+1$. Let H' be a hole induced by the node set $V(P^F_{S_iS_{i+1}}) \cup \{i, i+1\}$. Suppose that H' is not clean. Let u be a node of $G \backslash H'$ that has a neighbor in H' but is not of type W or S w.r.t. H'. Vertex u cannot see both $i, i+1$, as this implies $u \in W_i$ and component F cannot see both W_i and W_1, with $i \neq 1$. In addition, u cannot see one vertex of $\{i, i+1\}$, as this implies $u \in S_i \cup S_{i+1}$, which contradicts the minimality of $P^F_{S_iS_{i+1}}$. So u sees at least three nodes in $P^F_{S_iS_{i+1}}$. Note that $u \in F$ also contradicts the minimality of $P^F_{S_iS_{i+1}}$. Now because F sees W_1, S_i and S_{i+1} we have $u \in W_1$. But then a subset of the node set $V(H') \cup \{u, 2, \ldots, i-1\}$ induces a 3PC(u, i). Therefore H' is clean. Note that $x \in G \backslash (H' \cup W' \cup S')$. If $y \in G \backslash (H' \cup W' \cup S')$ or $y \in W'$, then our assumption is contradicted. So $y \in S'$. Let y_1 be the unique neighbor of y in H'. By Claim 10.1, y_1 is not adjacent to i. Hence the node set $V(H') \cup \{y, 2, \ldots, i-1\}$ induces a 3PC(y_1, i).

Now to prove the claim suppose that there exist components F and F' of $G \backslash (H \cup W \cup S)$ such that F (resp. F') sees y and S_i (resp. S_j), $i,j \neq 1,2$ and $i \neq j$. W.l.o.g. $i < j$. By the first observation $i,j \neq n$. By the second observation S_i (resp. S_j) does not see F' (resp F). In particular $F \neq F'$. Let s_i (resp. s_j) be a node of S_i (resp. S_j) that has a neighbor in F (resp. F'). By Lemma 3.2, s_i does not see s_j and hence the node set $V(P^F_{ys_i}) \cup V(P^{F'}_{ys_j}) \cup \{2, \ldots, i, \ldots, j\}$ induces a 3PC(y, i). This completes the proof of Claim 10.5. □

Claim 10.6. *If F is a component of $G \backslash (H \cup W \cup S)$ that sees y and S_i, $i \neq 1,2,n$, then it is not possible that two nonadjacent nodes of S_i both see F.*

Proof. Assume otherwise and let P be a shortest path in $F \cup S_i$ whose endnodes are nonadjacent nodes s, s' of S_i. Let H' be the hole induced by the node set $V(P) \cup \{i\}$. Suppose that H' is not clean. Let a be a node of $G \backslash H'$ that has a neighbor in H' but is not of type W or S w.r.t. H'. Vertex a cannot see both s, s', as we are assuming no diamond and no 4-hole. Node a cannot see i, since otherwise the minimality of P is contradicted. So a sees at least three nodes in P. Note that $a \in F$ also contradicts the minimality of P. By Claim 10.5, F does not see $(W \cup S)\backslash(W_1 \cup S_i)$, hence $a \in W_1$. But then a subset of the node set $V(H') \cup \{a, i+1, \ldots, n, 1\}$ induces a 3PC(a, i). Hence H' is clean. If $y \in G\backslash(H' \cup W' \cup S')$ or $y \in W'$, then H' contradicts our assumption. Hence $y \in S'$. Let y_1 be the unique neighbor of y in H'. By Claim 10.1, y_1 is not adjacent to i. Hence the node set $V(H') \cup \{y, i+1, \ldots, n, 1\}$ induces a 3PC(y_1, i). This completes the proof of Claim 10.6. □

Claim 10.7. *If* (i) *and* (ii) *of Claim* 10.3 *do not hold, then there is a component F of $G\backslash(H \cup W \cup S)$ that sees S_3 and y and contains a simplicial extreme of G that is in $G\backslash(N(x) \cup N(y))$.*

Proof. Suppose that (i) and (ii) of Claim 10.3 do not hold. Then there is a component F of $G\backslash(H \cup W \cup S)$ that sees S_3 and y. By Lemma 3.3, F does not see $(W \cup S)\backslash(W_1 \cup S_2 \cup S_3 \cup S_4)$. By Claim 10.5, F does not see S_4. Let G' be the subgraph of G induced by the node set $F \cup W_1 \cup S_2 \cup S_3 \cup \{x, 2, 3\}$. Since the theorem holds for G' and G' is not triangulated, there exist two nonadjacent simplicial extremes of G', say a and b, that are in $G'\backslash(N(2) \cup N(y))$. If a or b is in F we are done, so assume that $a, b \in S_3$.

As in the proof of Case 3 of Claim 10.3, there is a component F' (resp. F'') of $G\backslash(H \cup W \cup S)$ that sees a (resp. b) and y. By Claim 10.6, a (resp. b) does not see F'' (resp. F'). Hence the node set $V(P_{ya}^{F'}) \cup V(P_{yb}^{F''}) \cup \{2, 3\}$ induces a 3PC$(y, 3)$. This completes the proof of Claim 10.7. □

Claim 10.8. *If* (i) *and* (ii) *of Claim* 10.4 *do not hold, then there is a component F of $G\backslash(H \cup W \cup S)$ that sees S_{n-1} and y and contains a simplicial extreme of G that is in $G\backslash(N(x) \cup N(y))$.*

Proof. Suppose that (i) and (ii) of Claim 10.4 do not hold. Then there is a component F of $G\backslash(H \cup W \cup S)$ that sees S_{n-1} and y. By Lemma 3.3, F does not see $(W \cup S)\backslash(W_1 \cup S_n \cup S_{n-1} \cup S_{n-2})$. By Claim 10.5, F does not see $S_n \cup S_{n-2}$. Let G' be the subgraph of G induced by the node set $F \cup W_1 \cup S_{n-1} \cup \{1, 2, n-1, n\}$. Since the theorem holds for G' and G' is not triangulated, there exist two nonadjacent simplicial extremes of G', say a and b, that are in $G'\backslash(N(x) \cup N(y))$. If a or b is in F we are done, so assume that $a, b \in S_{n-1}$.

As in the proof of Case 3 of Claim 10.4, there is a component F' (resp. F'') of $G\backslash(H \cup W \cup S)$ that sees a (resp. b) and y. By Claim 10.6, a (resp. b) does not see F'' (resp. F'). Hence the node set $V(P_{ya}^{F'}) \cup V(P_{yb}^{F''}) \cup \{1, n, n-1\}$ induces a 3PC$(y, n-1)$. This completes the proof of Claim 10.8. □

To prove the claim we now consider the following three cases.

Case 1: Condition (i) of Claim 10.3 holds. If (i) of Claim 10.4 holds, then $(*)$ holds. If (ii) of Claim 10.4 holds, then $(*)$ holds by Claim 5. If (i) and (ii) of Claim 10.4 do not hold, then $(*)$ holds by Claim 10.8.

Case 2: Condition (ii) of Claim 10.3 holds. Let F be a component of $G\backslash(H\cup W\cup S)$ that sees $W_2\cup S_3\cup W_3$ and does not see y. By Claim 5, F contains a simplicial extreme of G. If (i) of Claim 10.4 holds, then (∗) holds. Suppose that (ii) of Claim 10.4 holds and let F' be a component of $G\backslash(H\cup W\cup S)$ that sees $W_{n-2}\cup S_{n-1}\cup W_{n-1}$ and does not see y. We may assume that $F=F'$, else (∗) holds by Claim 5. By Lemma 3.3, F sees both W and S. W.l.o.g. suppose that F sees S_3 and W_{n-1}. Let H' be a hole induced by the node set $V(P^F_{S_3W_{n-1}})\cup\{3,\dots,n-1\}$. By Claim 10.1, y does not see H'. H' is not clean, else our assumption is contradicted. By Lemma 3.6, there is a bug (H',u). Since $u\in W\cup S$, by Claim 10.1, y does not see u. Hence (H',u) is a bug that contradicts our assumption. Finally, if (i) and (ii) of Claim 10.4 do not hold, by Claim 10.8, there is a component F' of $G\backslash(H\cup W\cup S)$ that sees S_{n-1} and y and contains a simplicial extreme of G that is in $G\backslash(N(x)\cup N(y))$. Since $F\neq F'$, (∗) holds.

Case 3: Conditions (i) and (ii) of Claim 10.3 do not hold. Then by Claim 10.7, there is a component F of $G\backslash(H\cup W\cup S)$ that sees S_3 and y and contains a simplicial extreme of G that is in $G\backslash(N(x)\cup N(y))$. So if (i) of Claim 10.4 holds, then (∗) holds. If (ii) of Claim 10.4 holds, then (∗) holds by Claim 5. By Claim 10.5, it is not possible that (iii) of Claim 10.4 holds.

This completes the proof of Claim 10. □

Claim 11. *There exists a clean hole H such that either $x,y\in H$ or $x\in H$ and $y\in W$.*

Proof. Let $N_{xy}=N(x)\cap N(y)$, $N_x=N(x)\backslash(N_{xy}\cup\{y\})$ and $N_y=N(y)\backslash(N_{xy}\cup\{x\})$. If $N_x=\emptyset$, then by Lemma 2.3, x is a simplicial vertex of G, which contradicts Claim 1. So $N_x\neq\emptyset$ and similarly $N_y\neq\emptyset$. By Lemma 2.3, $C=N_{xy}\cup\{x,y\}$ is a clique. By Claim 1, C cannot be a cutset, so $G\backslash C$ is connected. Let $P=p_1,\dots,p_k$ be a shortest path in $G\backslash C$ such that $p_1\in N_x$ and $p_k\in N_y$. Let H be a hole induced by the node set $V(P)\cup\{x,y\}$. We may assume that H is not clean, since otherwise we are done. Let u be a node of $G\backslash H$ that has a neighbor in H but is not of type W or S w.r.t. H. By the choice of H, $u\in N_{xy}$. Let p_i be the node of P with smallest index that is adjacent to a node of N_{xy}. W.l.o.g. assume that p_i is adjacent to u. By Lemma 2.3, $i>1$, and so $H'=x,p_1,\dots,p_i,u,x$ is a hole. By the choice of p_i, no node of $N_{xy}\backslash\{u\}$ has a neighbor in $p_1,\dots,p_{i-1}$. Since G contains no diamond, no node of $N_{xy}\backslash\{u\}$ sees p_i. Hence H' is clean. Since $x\in H'$ and $y\in W'$, this completes the proof of Claim 11.

By Claims 8, 10 and 11, (∗) holds and the proof of the theorem is complete. □

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