BIGRAPHS: coordinating mobile computation

March 2006

Robin Milner

including joint work with **Jamey Leifer** and **Ole Jensen**

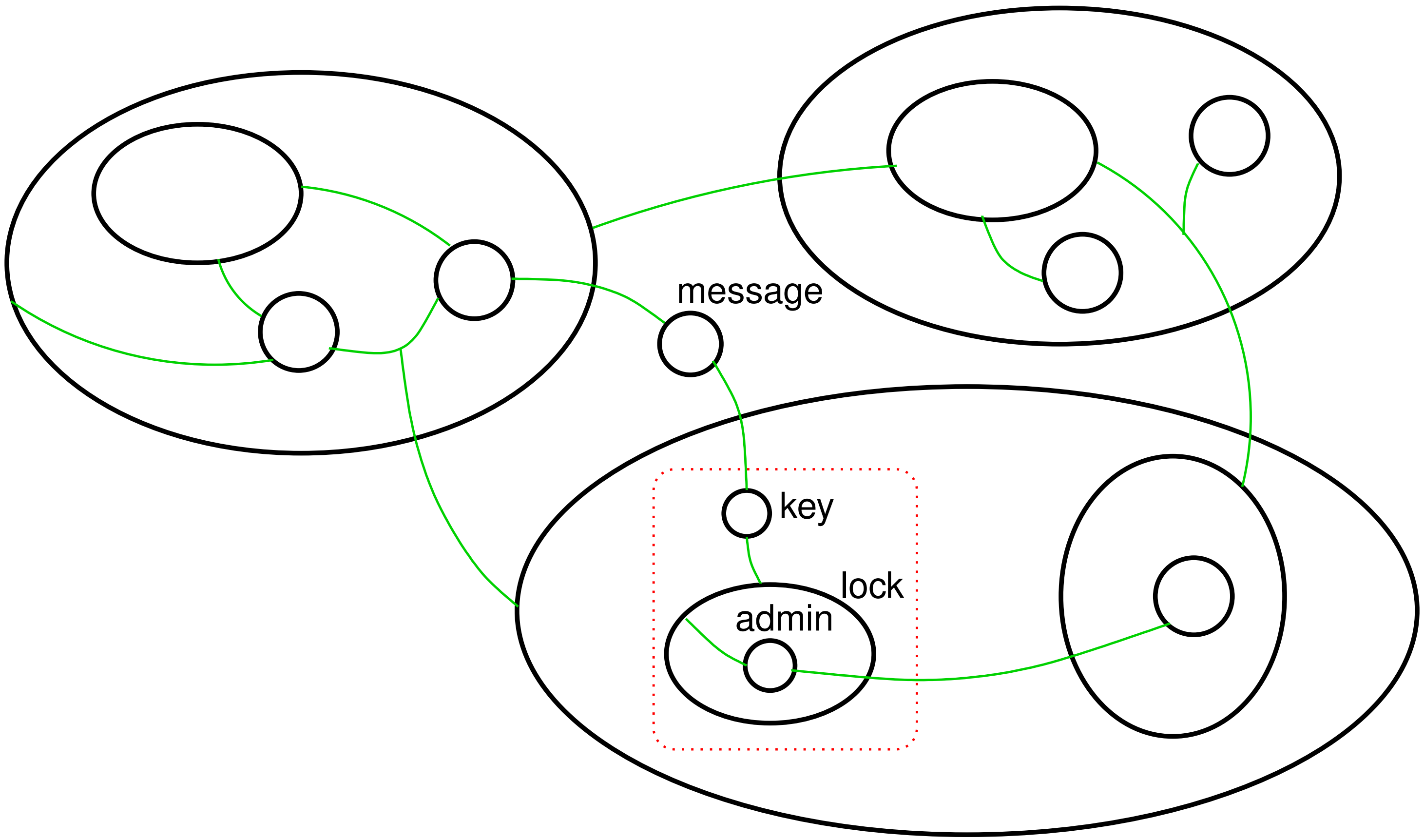
Part I Examples of bigraph reactions

Part II S-categories, reactions and transitions

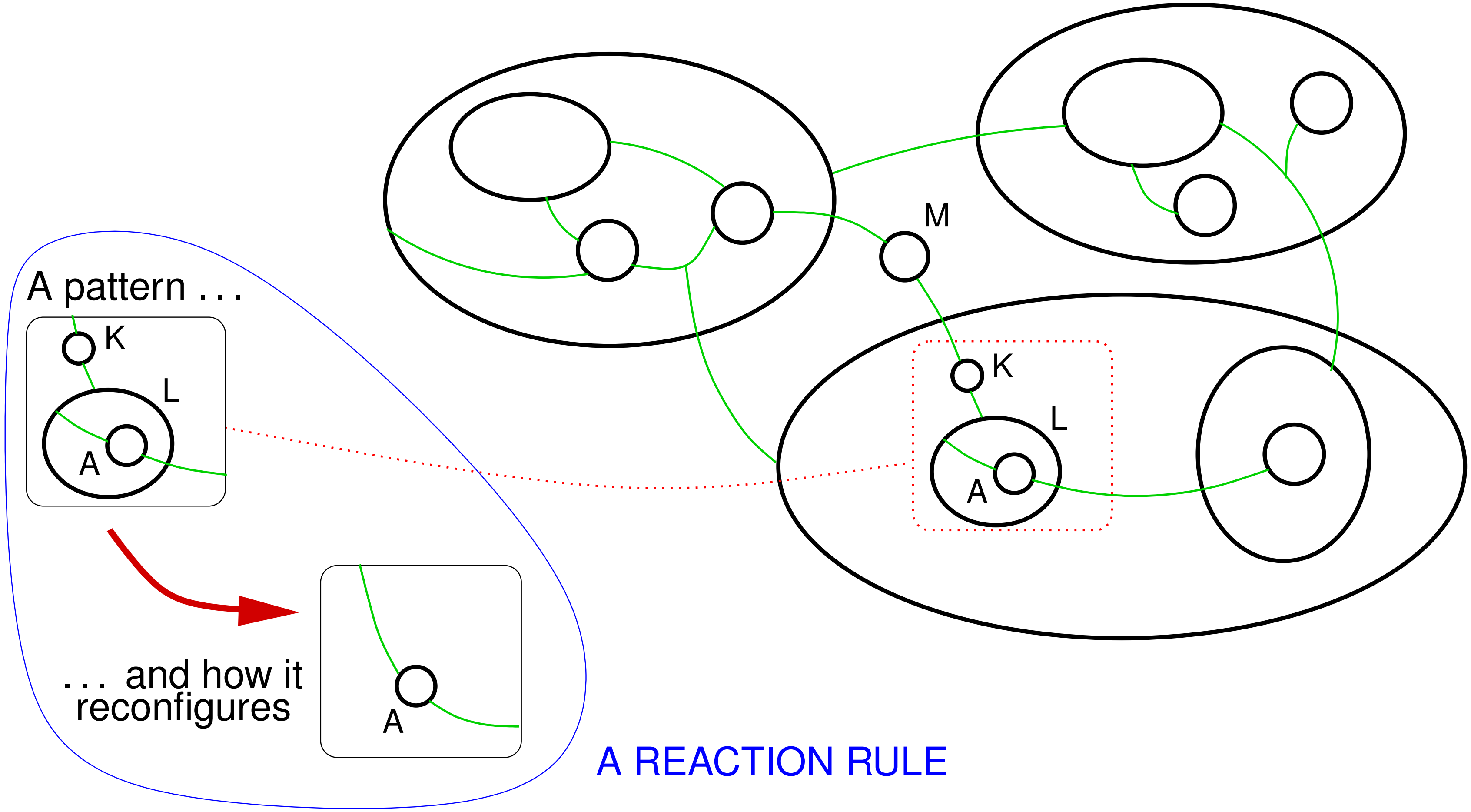
Part III Link graphs, sorting and Petri nets

Part I EXAMPLES OF BIGRAPH REACTIONS

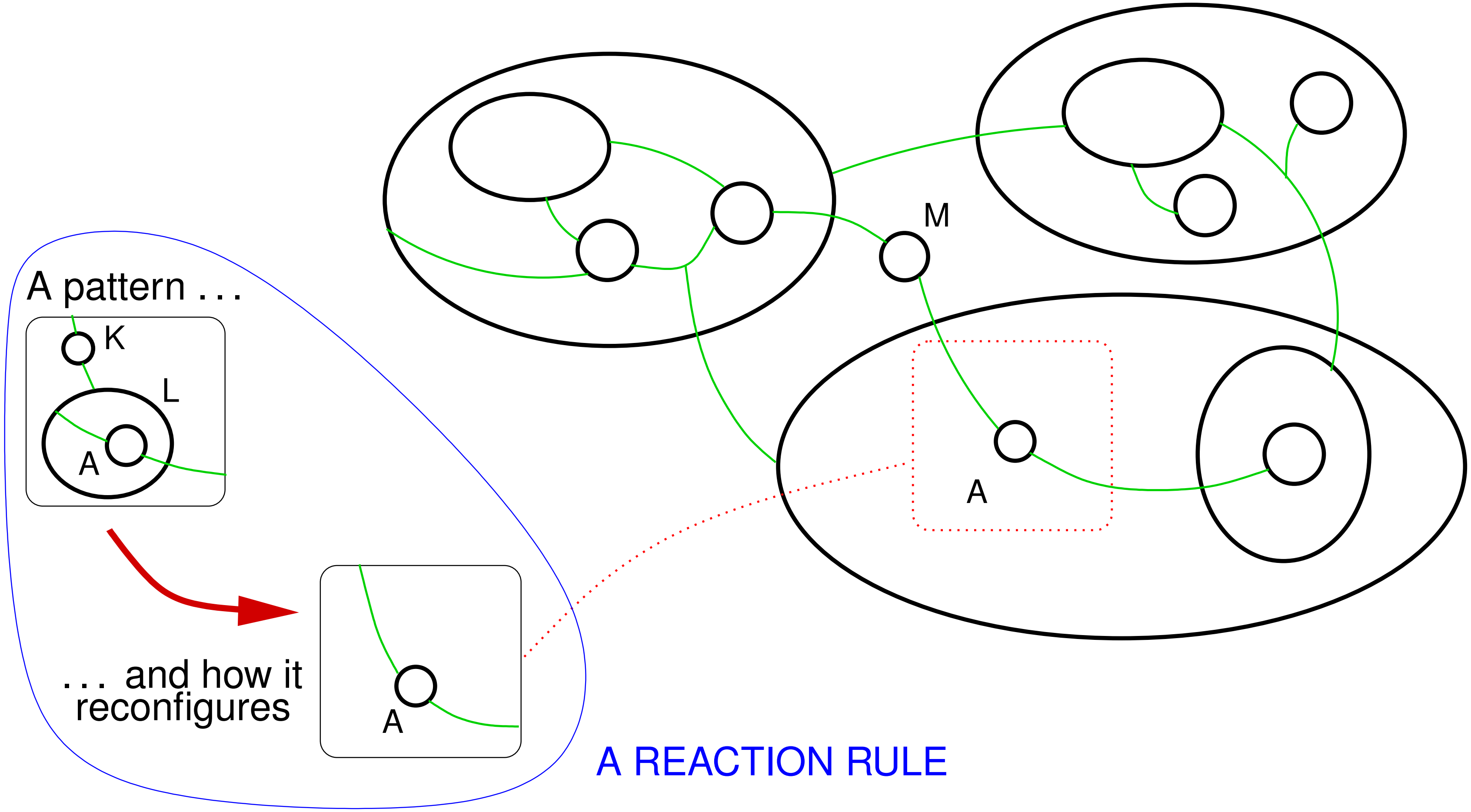
A TYPICAL BIGRAPH



HOW A SYSTEM MAY RECONFIGURE . . . . . .



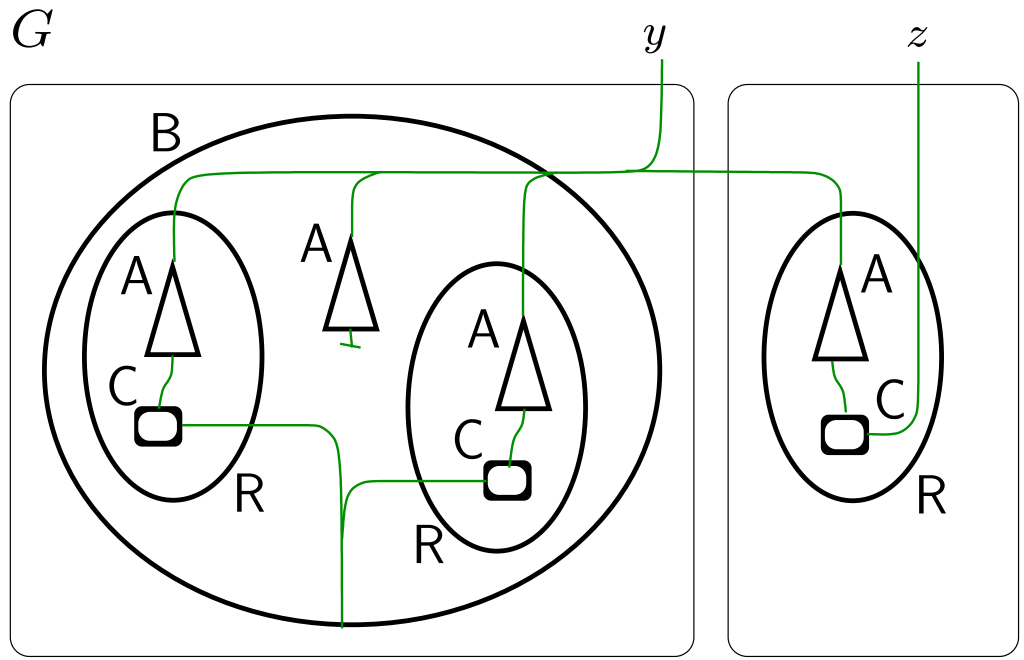
. . . . . . AND THE NEW CONFIGURATION



INTERACTIONS IN A BUILT ENVIRONMENT (1)

A bigraph G with two regions, representing a conference call

B	BUILDING
R	ROOM
A	AGENT
C	COMPUTER



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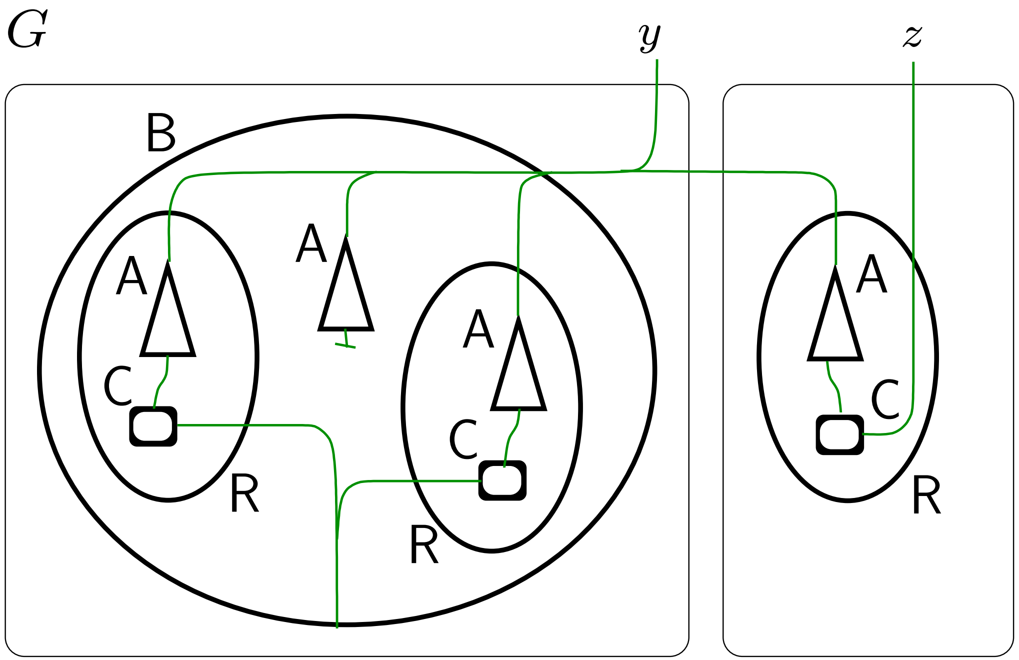
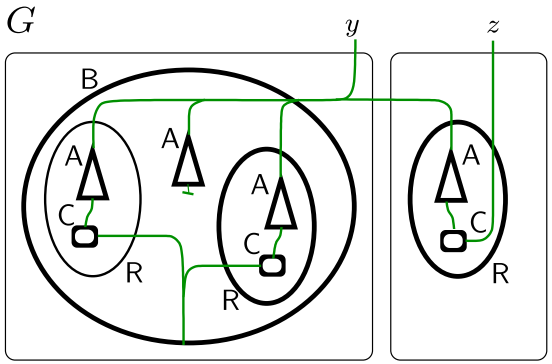


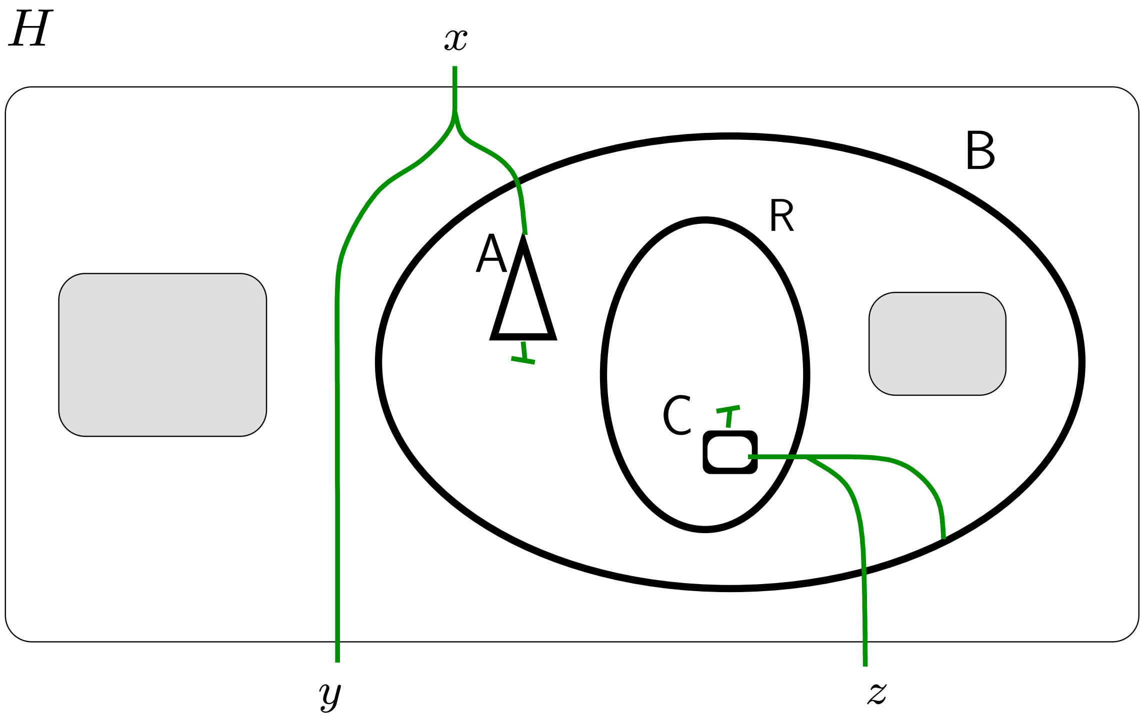
algebraic form:

$$/x\ (\mathsf{B}x(\mathsf{R}(/u\ \mathsf{A}yu \mid \mathsf{C}ux) \mid /u\ \mathsf{A}yu)\mathsf{R}(/u\ \mathsf{A}yu \mid \mathsf{C}ux))$$
$$\parallel \mathsf{R}(/v\ \mathsf{A}yv \mid \mathsf{C}vz)$$

INTERACTIONS IN A SENTIENT BUILDING (2)

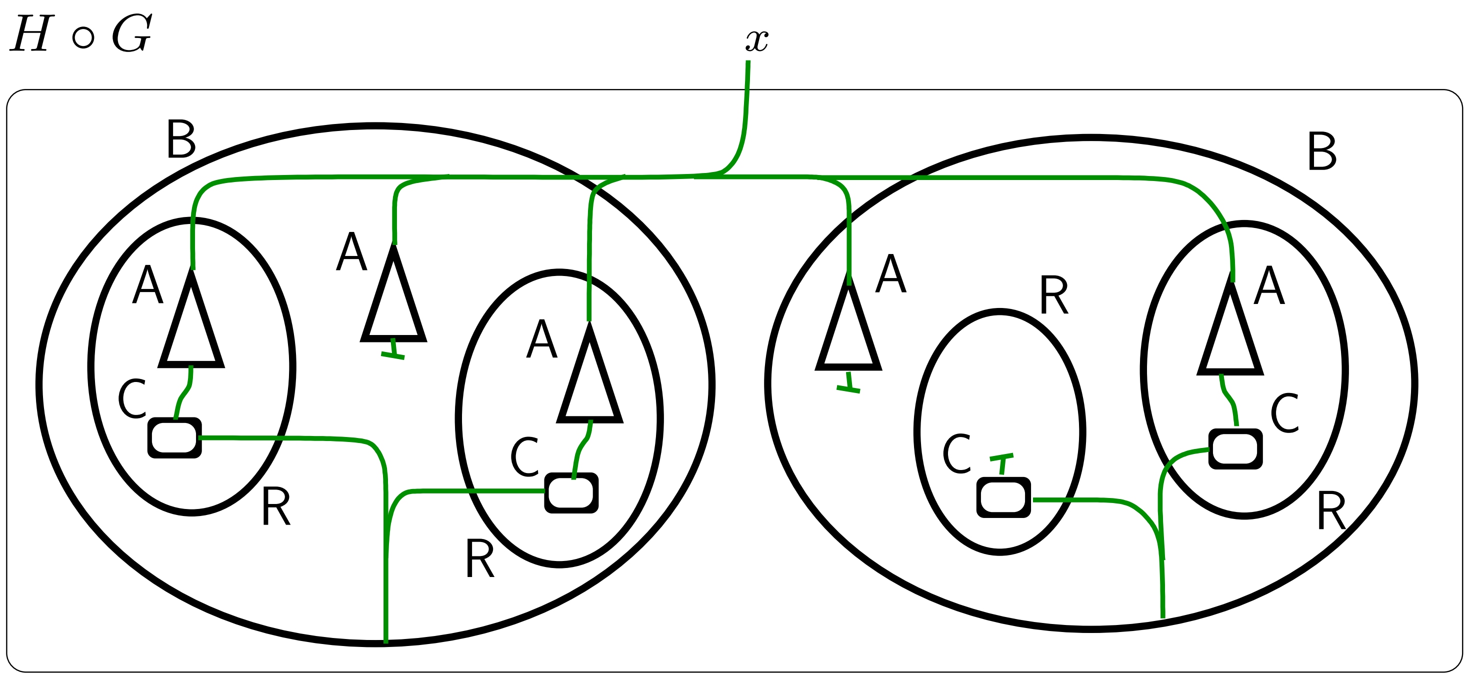
A host environment H,
which G may inhabit





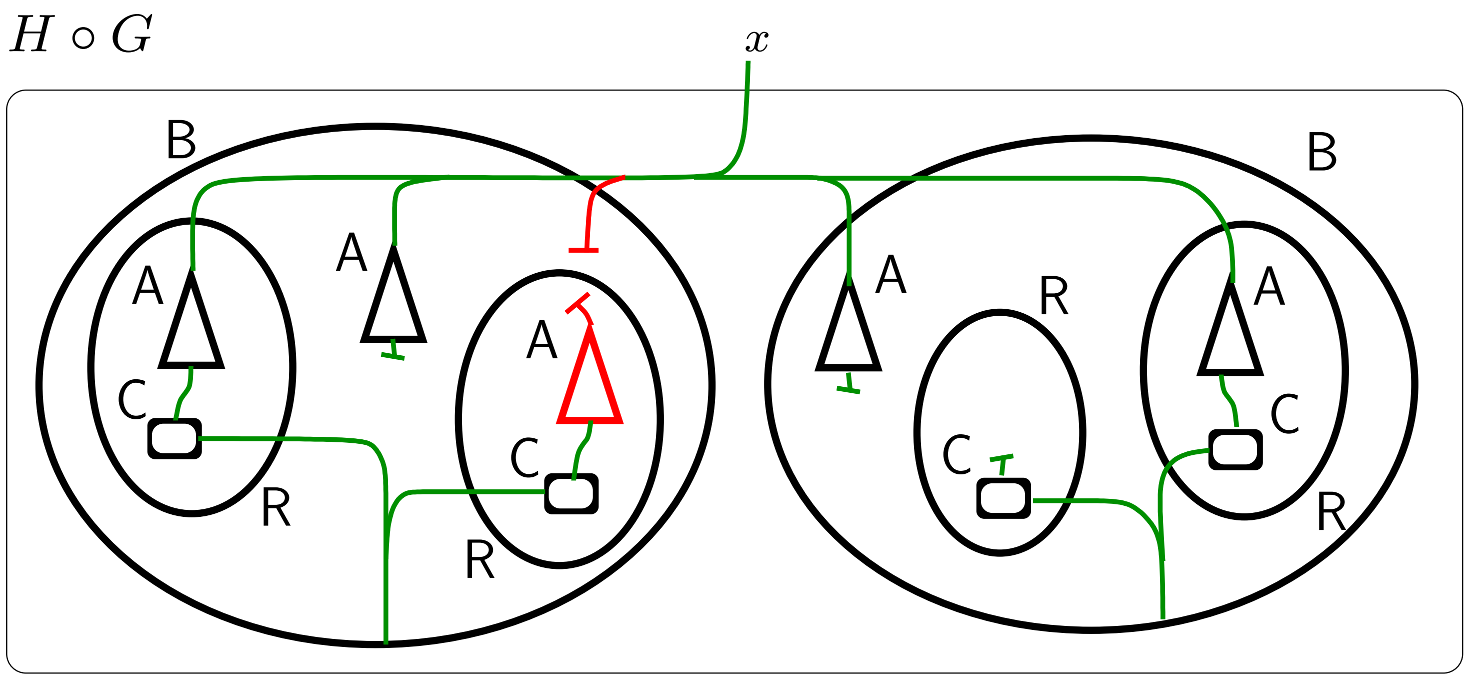
INTERACTIONS IN A SENTIENT BUILDING (3)

The larger environment, $H \circ G$.



INTERACTIONS IN A SENTIENT BUILDING (4)

One agent leaves the call!

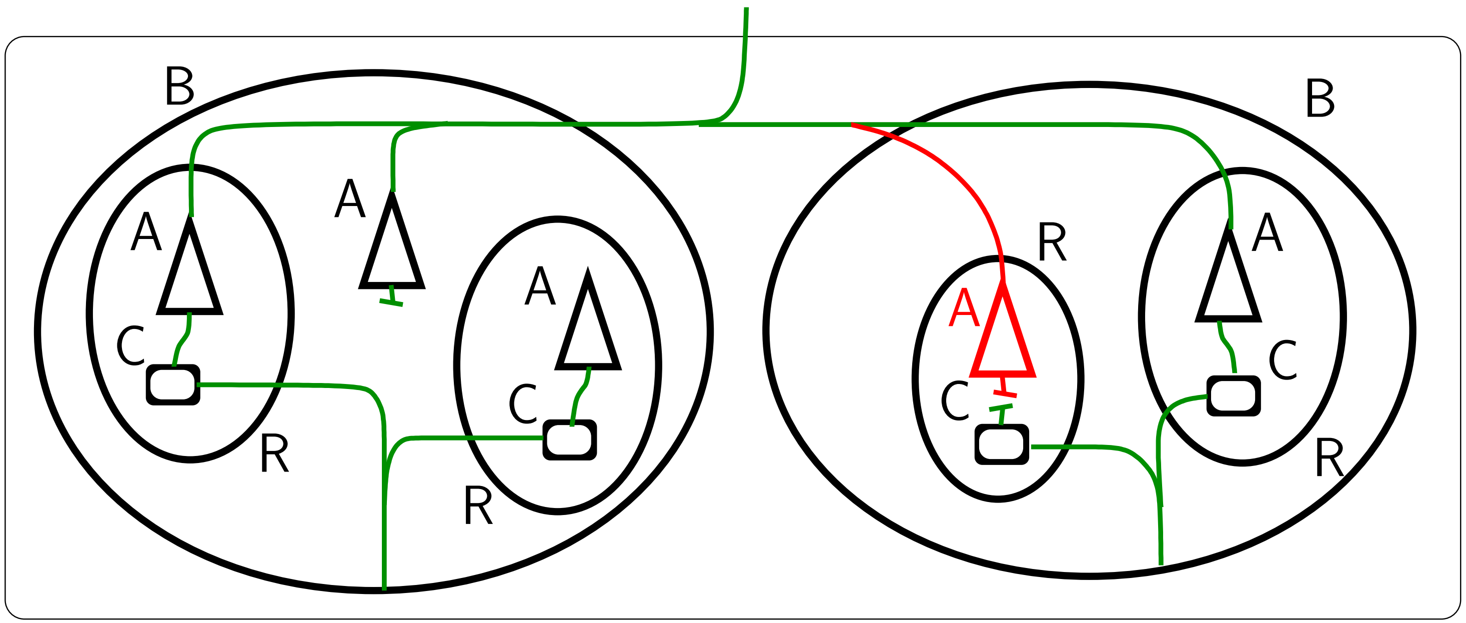


INTERACTIONS IN A SENTIENT BUILDING (5)

One agent leaves the call

Another moves into a room . . .

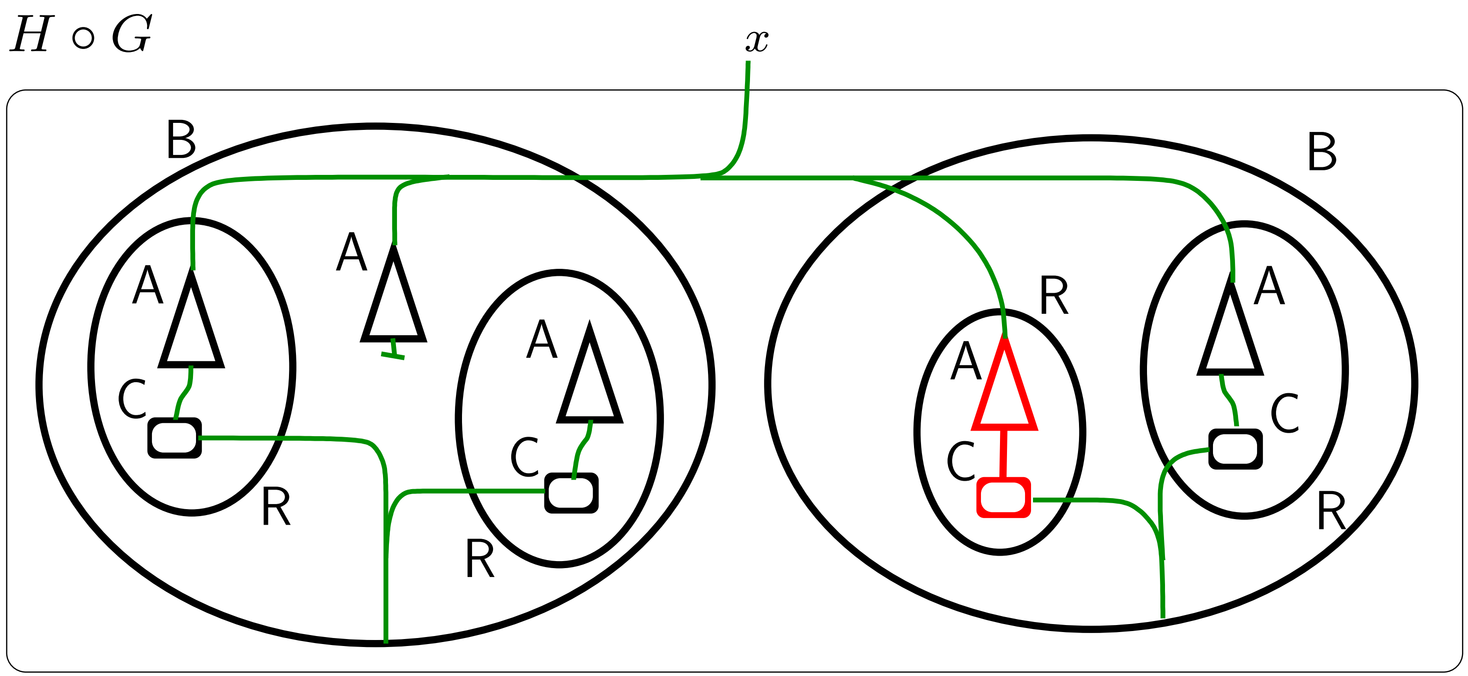
$H \circ G$



INTERACTIONS IN A SENTIENT BUILDING (6)

The larger environment, $H \circ G$.

One agent leaves the call
Another moves into a room . . .
. . . and is sensed and logged in!



SIGNATURES AND CONTROLS

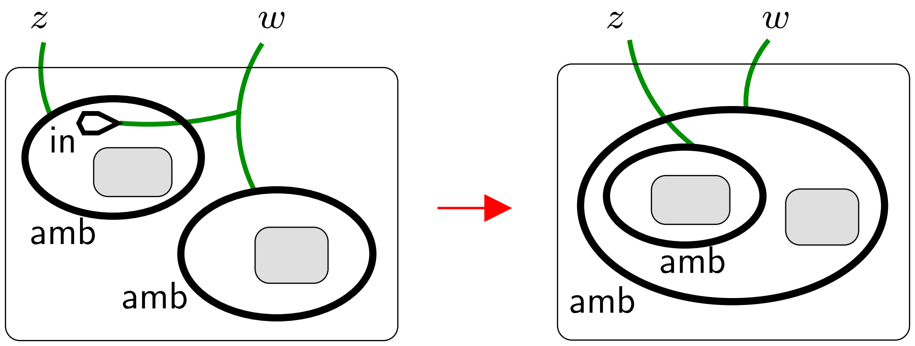
A **signature** $\mathcal{K}$ determines a set of **controls** $\mathsf{K}, \mathsf{L}, \ldots$.
In a bigraph over $\mathcal{K}$, each node is assigned a control.

$\mathcal{K}$ also determines, for each control K:

- an **arity** n: the number of **ports**
- whether **atomic**; if not, whether **active** or **passive**

Atomic nodes are empty. Active nodes permit reaction inside.

A REACTION RULE FOR THE AMBIENT CALCULUS



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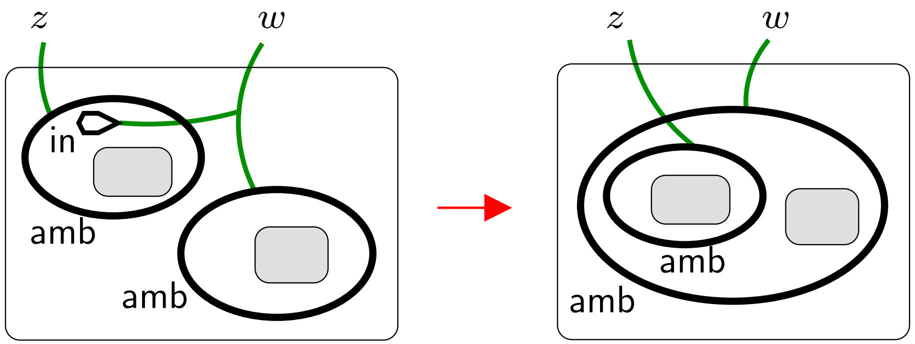

$$\mathsf{amb}_z.(\mathsf{in}_w \mid d) \mid \mathsf{amb}_w.e \longrightarrow \mathsf{amb}_w.(\mathsf{amb}_z.d \mid e)$$

$\mathsf{amb} : 1$ is *non-atomic* and *active*

$\mathsf{in} : 1$ is *atomic*

A REACTION RULE FOR CCS

$$(\overline{x}.P + M) \,|\, (x.Q + N) \;\longrightarrow\; P \,|\, Q$$

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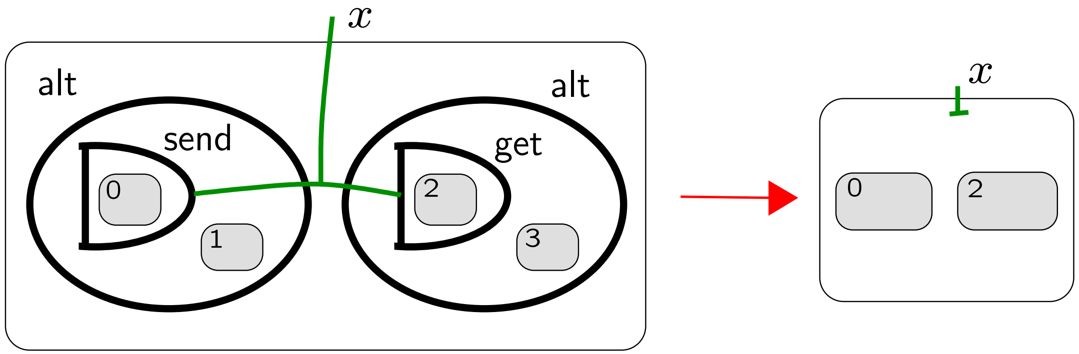

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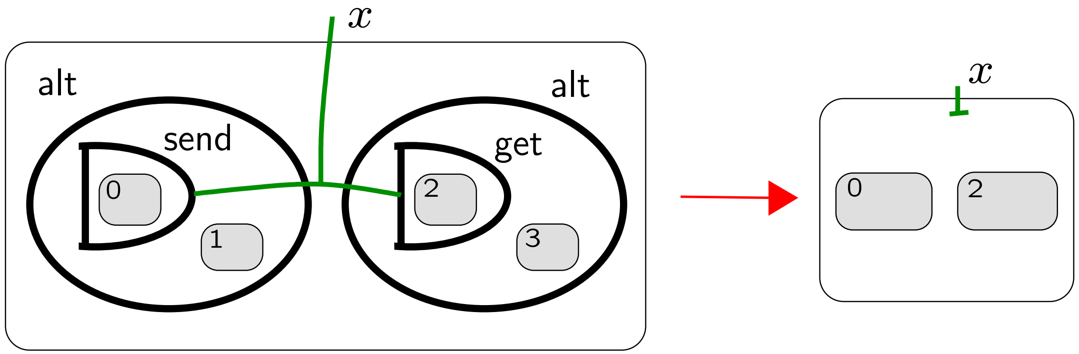


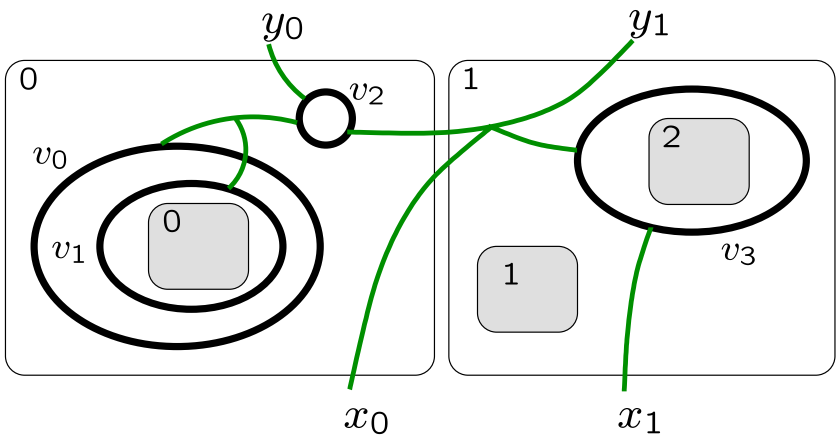
$$\mathsf{alt}.\,(\mathsf{send}_x.p \,|\, m) \;|\; \mathsf{alt}.\,(\mathsf{get}_x.q \,|\, n) \;\longrightarrow\; x \,|\, p \,|\, q$$

alt, send, get are *non-atomic* and *passive*

RESOLVING A BIGRAPH INTO PARTS

bigraph

$G\colon \langle m, X\rangle \to \langle n, Y\rangle$



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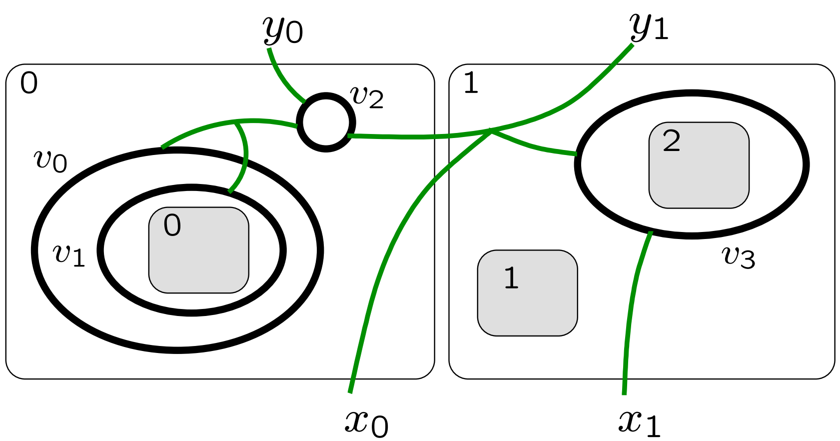
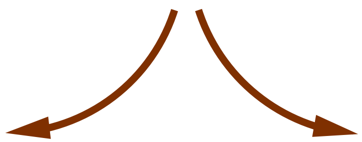
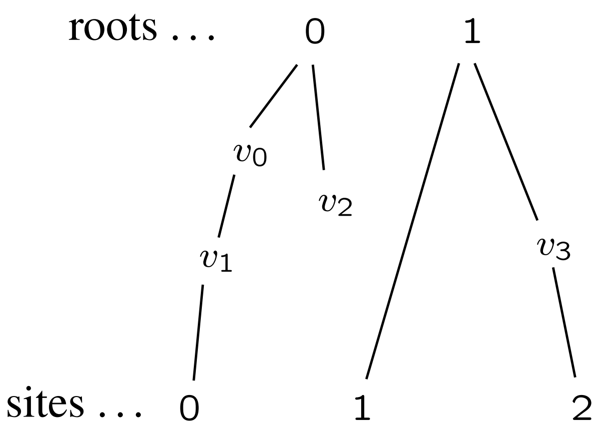
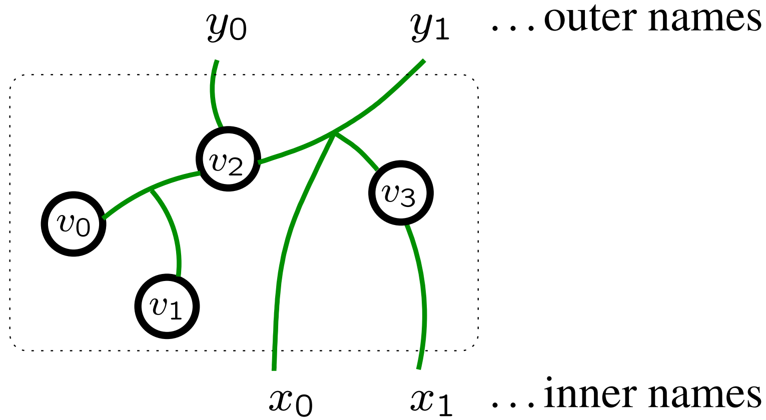


place graph
$G^{\mathrm{P}}\colon m \to n$

link graph
$G^{\mathrm{L}}\colon X \to Y$



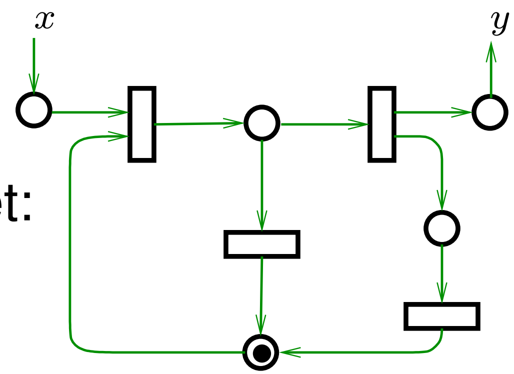




PETRI NETS AS LINK GRAPHS

We consider **condition-event nets** with named conditions

A c-e net:



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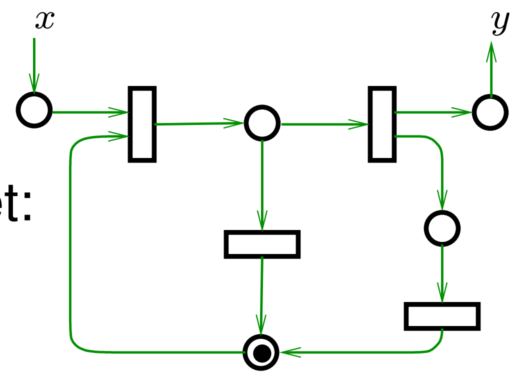
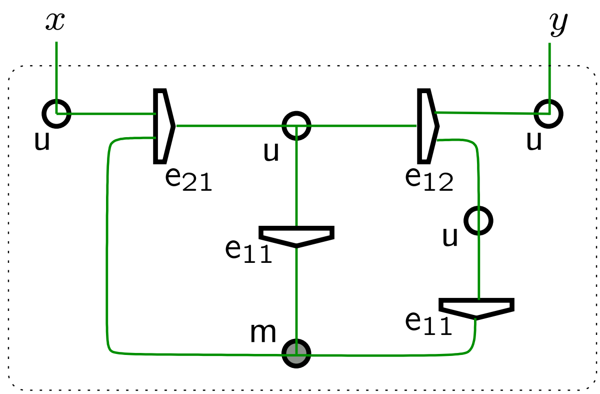


its link graph:

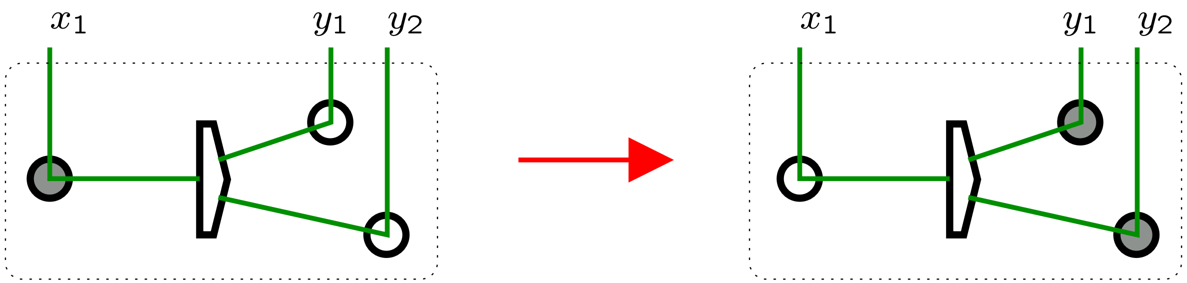


m : 1	marked condition	
u : 1	unmarked condition	
e_{hk} : $h + k$	event with	h pre-conditions
		k post-conditions

REACTION RULES FOR PETRI NETS

An event can e_{hk} 'fire' iff its preconditions are all marked and its postconditions all unmarked

The rule for $h = 1$, $k = 2$:



Part II S-CATEGORIES, REACTIONS and TRANSITIONS

S-CATEGORIES

An **s-category** **´A** :

- is like a **category**: it has *objects* $I, J, \ldots$ (here called *interfaces*), and arrows $A : I \to J$ between them. Under certain conditions we can *compose* $A : I \to J$ and $B : J \to K$ to form $B \circ A : I \to K$.

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- is **symmetric monoidal**: given suitably disjoint interfaces I, J we can form $I \otimes J$, their *tensor product*.
Given $A : I \to I'$ and $B : J \to J'$ we can form $A \otimes B : I \otimes J \to I' \otimes J'$, their *tensor product*; it is often merely *juxtaposition*.

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Certain axioms hold for $\circ$ and $\otimes$. For this talk, the difference between categories and s-categories can mostly be ignored.

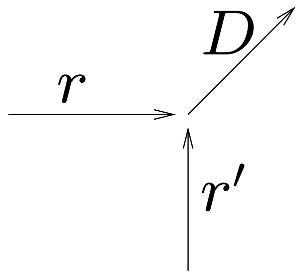
REACTIVE SYSTEM ´A(´$\mathcal{R}$)

The empty interface ϵ, the unit of $\otimes$, is called the **origin**.
A **ground** arrow is $a : \epsilon \rightarrow I$, written $a : I$.

To form ´**A**(´$\mathcal{R}$), supply ´**A** with a set ´$\mathcal{R}$ of **ground reaction rules** $(r : I, r' : I)$.

Reaction relation: the smallest $\longrightarrow\!\!\triangleright$ such that

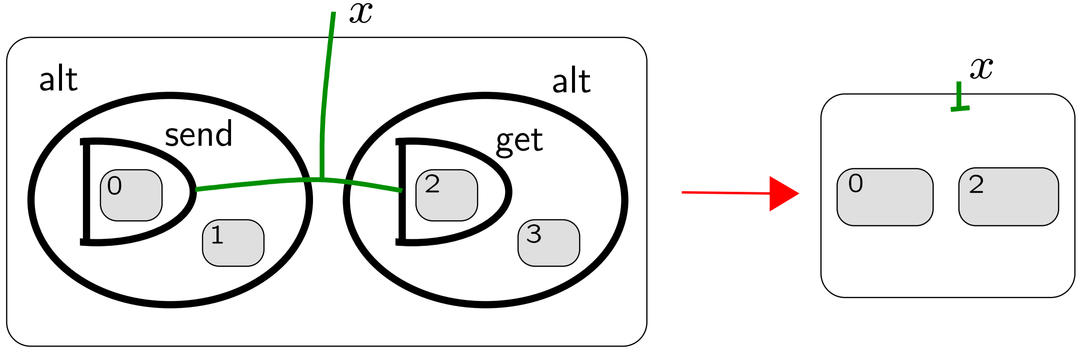
$$D \circ r \longrightarrow\!\!\triangleright D \circ r'$$



for all contexts D and ground rules (r, r').

The REACTION RULE for CCS is PARAMETRIC

$$(\overline{x}.P + M) \,|\, (x.Q + N) \;\longrightarrow\; P \,|\, Q$$



Parametric rules are defined using contexts

REACTION EQUIVALENCE:

Define **reaction equivalence** of ground bigraphs as a *bisimilarity*:

if $a \approx b$ and $a \longrightarrow\!\!\triangleright\, a'$, then $b \longrightarrow\!\!\triangleright\, b'$ such that $a' \approx b'$.

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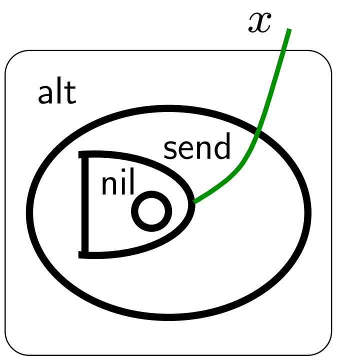
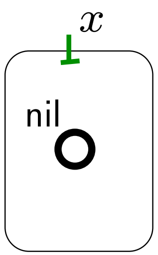


a



b

Is $a \approx b$ here? YES!!

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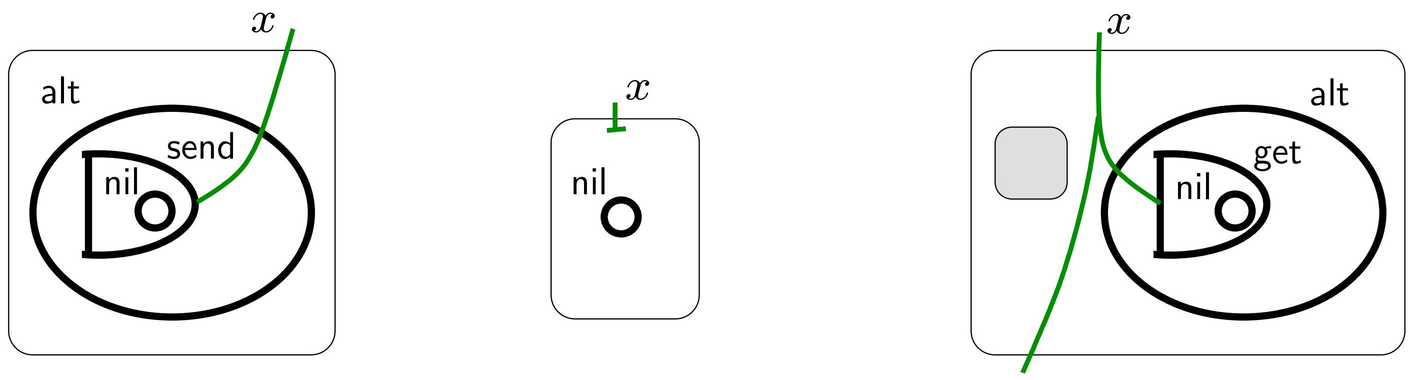

a b C

Is $a \approx b$ here? YES!! But they '**react with their environment**' differently; e.g. $C \circ a$ can react, but $C \circ b$ cannot. So $\approx$ is not a congruence.
We would like a congruence!

‘RAW’ LABELLED TRANSITION SYSTEM **´A($\mathcal{L}$)**

Equip **´A** with a **labelled transition system** $\mathcal{L} = (\mathsf{Ag}, \mathsf{Lab}, \mathsf{Trans})$ having **agents** Ag (ground arrows), **labels** Lab, and **transitions** Trans of form

$$a \xrightarrow{\ell} a'$$

where $a, a' \in \mathsf{Ag}$ and $\ell \in \mathsf{Lab}$.

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In a transition, think of ℓ as **observing** the agent a. This leads to equivalences such as **bisimilarity**, the largest symmetric relation $\sim$ such that

if $a \sim b$ *and* $b \xrightarrow{\ell} b'$, *then* $a \xrightarrow{\ell} a'$ *for some* $a' \sim b'$ *.*

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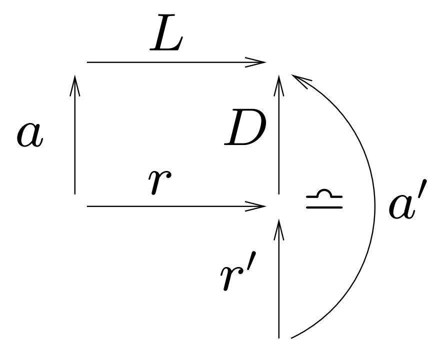
ANSWER: Contextual LTSs, when Relative Pushouts (RPOs) exist.

CONTEXTUAL TRANSITION SYSTEMS

Contextual transition for ´**A**(´$\mathcal{R}$): a transition $a \xrightarrow{L} a'$ where L and D are **contexts** making the diagram commute, for some rule (r, r') in ´$\mathcal{R}$.



Contextual transition system: $\mathcal{L} = (\mathsf{Ag}, \mathsf{Trans})$, where

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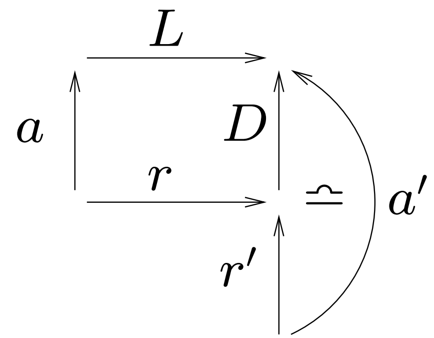

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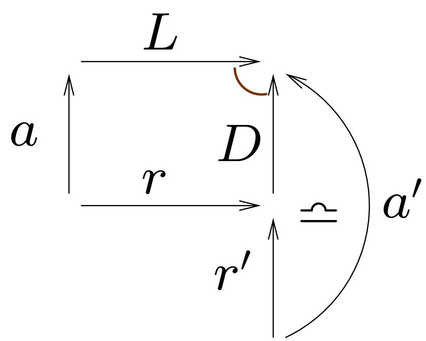
The **full** transition system has *all* possible transitions. But this is *too many*!
So consider only the transitions for which the pair (L, D) is minimal:

STANDARD TRANSITIONS AND BISIMILARITY



Standard contextual LTS $\mathsf{ST} = (\mathsf{Ag}_{\mathsf{ST}}, \mathsf{Trans}_{\mathsf{ST}})$:

- $\mathsf{Ag}_{\mathsf{ST}}$ has all ground arrows
- $\mathsf{Trans}_{\mathsf{ST}}$ has all **minimal** transitions.

Standard bisimilarity: the largest symmetric relation $\sim$ such that

if $a \sim b$ *and* $b \xrightarrow{L} b'$, *with* La *defined, then* $a \xrightarrow{L} a'$ *for some* $a' \sim b'$.

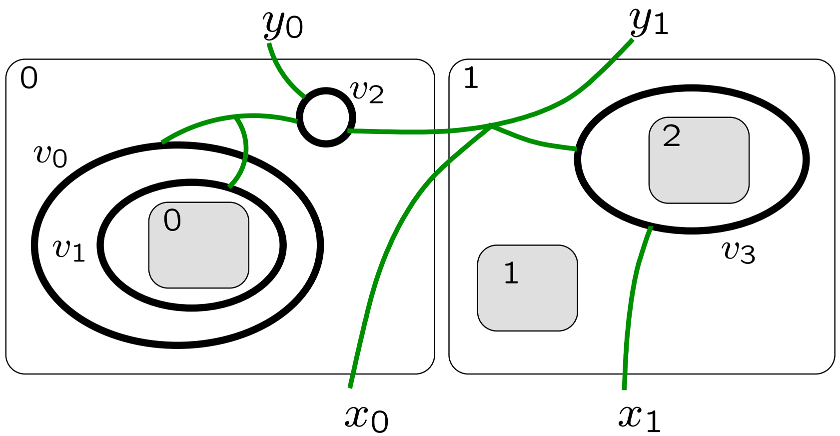
THEOREM: In any reactive system with RPOs, standard bisimilarity $\sim$ is a congruence. (*Leifer, Milner*)

Part III LINK GRAPHS, SORTING and PETRI NETS

RESOLVING A BIGRAPH INTO PARTS

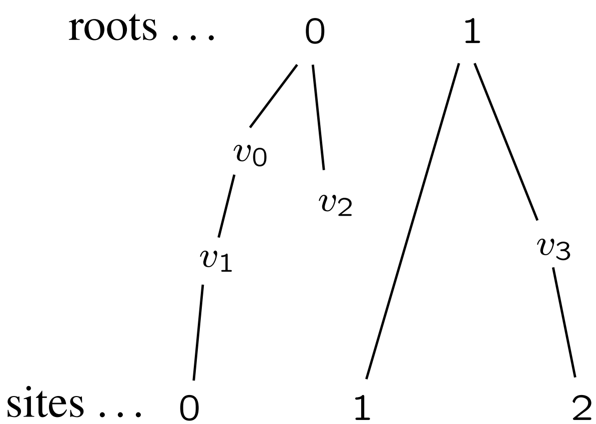
bigraph

$G\colon \langle m, X\rangle \to \langle n, Y\rangle$



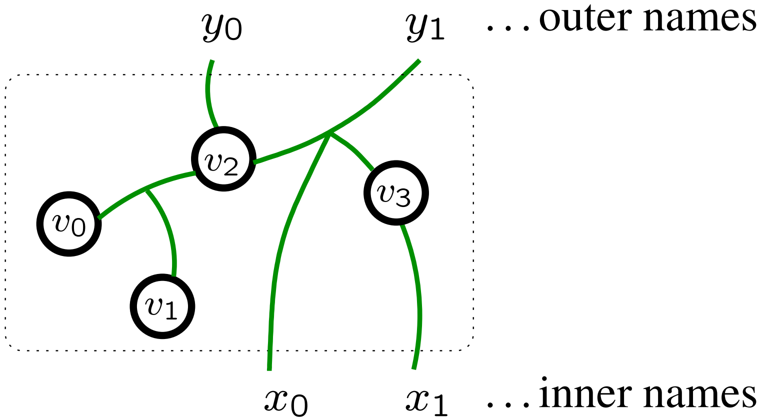
place graph

$G^{\mathrm{P}}\colon m \to n$



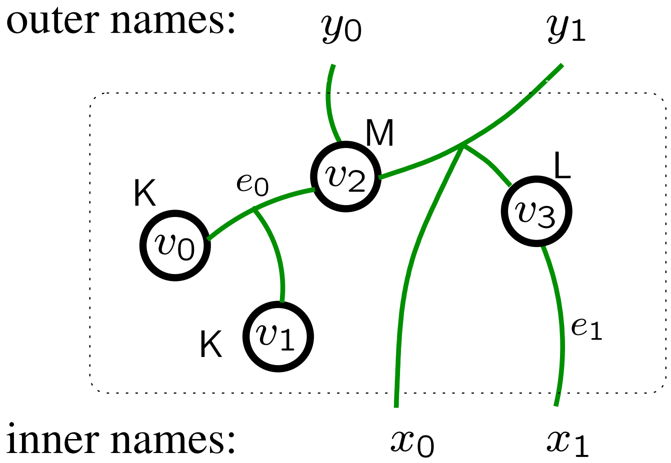
link graph

$G^{\mathrm{L}}\colon X \to Y$



A LINK GRAPH over signature $\mathcal{K} = \{\mathsf{K}, \mathsf{L}, \mathsf{M}\}$

$$G = (V, E, \mathit{ctrl}, \mathit{link}) \colon X \to Y$$



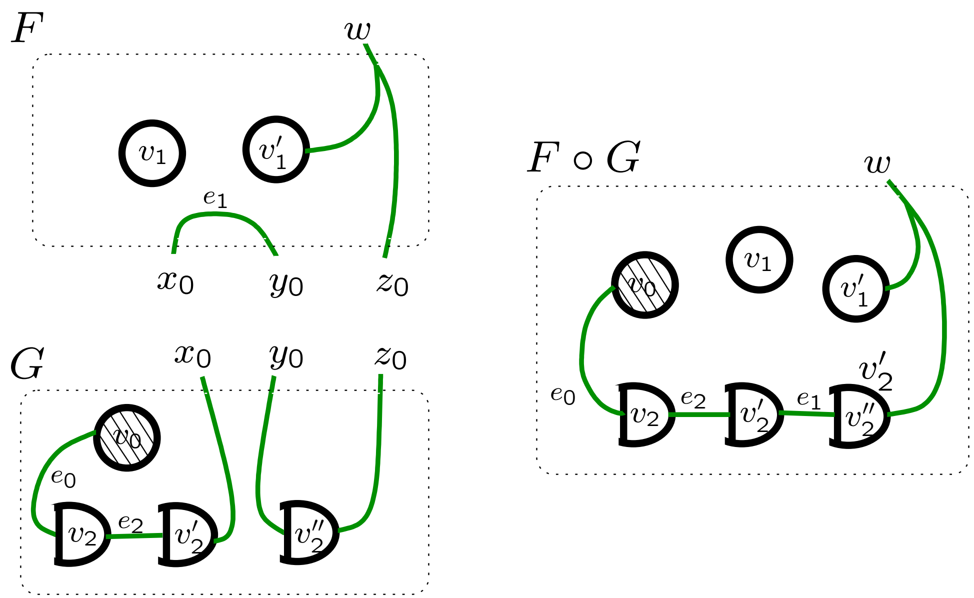
- **nodes** V **edges** E
- **control map** $\mathit{ctrl} : V \to \mathcal{K}$
- **ports** $P \stackrel{\text{def}}{=} \sum_{v \in V} \mathit{ar}(\mathit{ctrl}(v))$
- **link map** $\mathit{link} \colon X \uplus P \to E \uplus Y$

Inner names and ports are **points**.

Outer names are **open links**; edges are **closed links.**

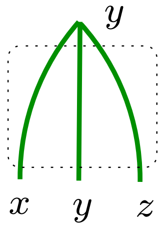
THE S-CATEGORY OF LINK GRAPHS

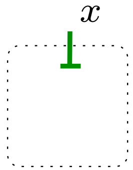
composition: link the names

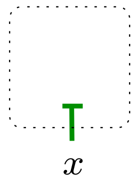


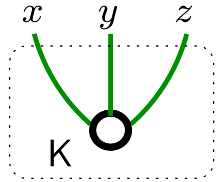
ELEMENTARY LINK GRAPHS AND PARALLEL PRODUCT

All link graphs can be built by $\circ$ and $\otimes$ from **elements**:

y/xyz 

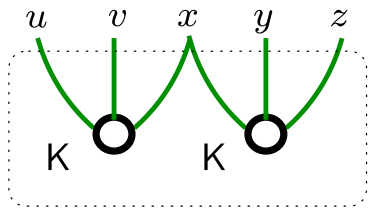
x 

$/x$ 

K_{xyz} 

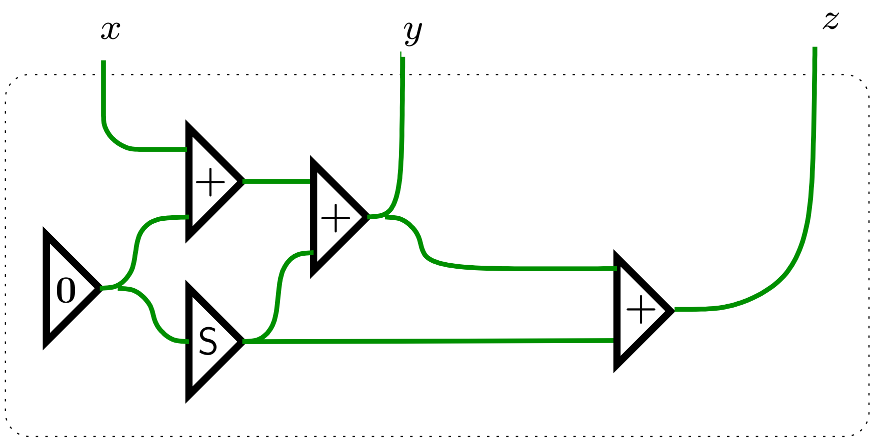
linkings (substitution **idle name** **closure)** **atoms**

Parallel product $F \mid G$ **:** derivable from tensor product and substitution.
Like $F \otimes G$, but outer names may be shared. For example:

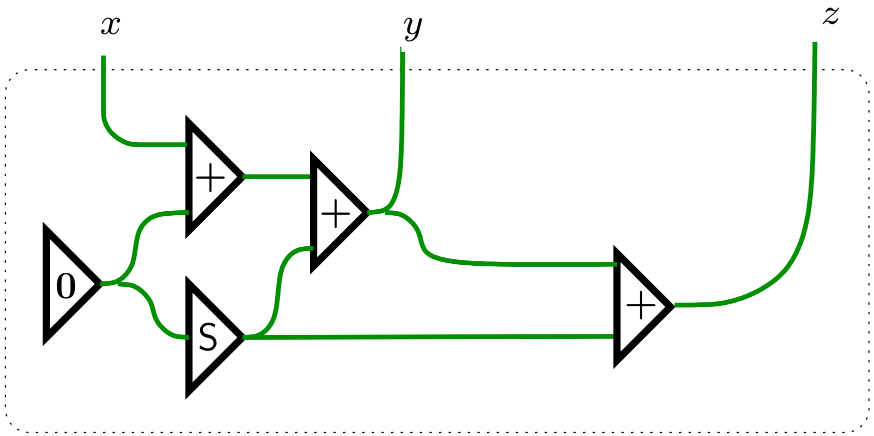
$\mathsf{K}_{uvx} \mid \mathsf{K}_{xyz}$ 

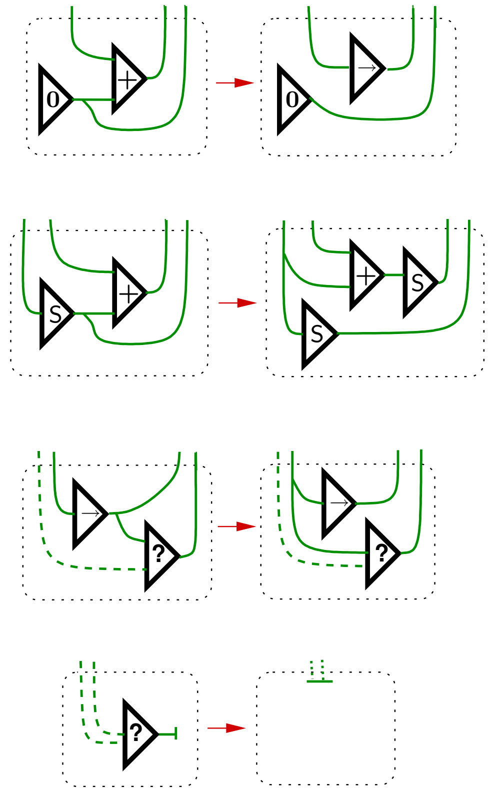
$= (x/wx) \circ (\mathsf{K}_{uvw} \otimes \mathsf{K}_{xyz})$

AN ARITHMETIC NET



AN ARITHMETIC NET and EVALUATION RULES





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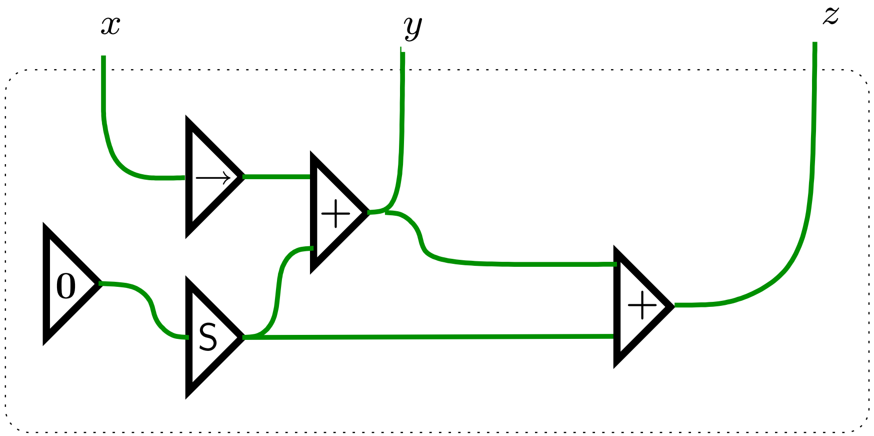

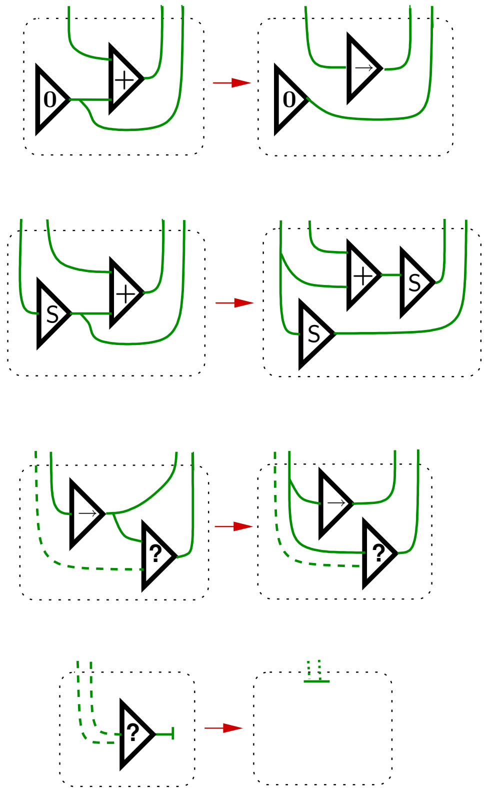

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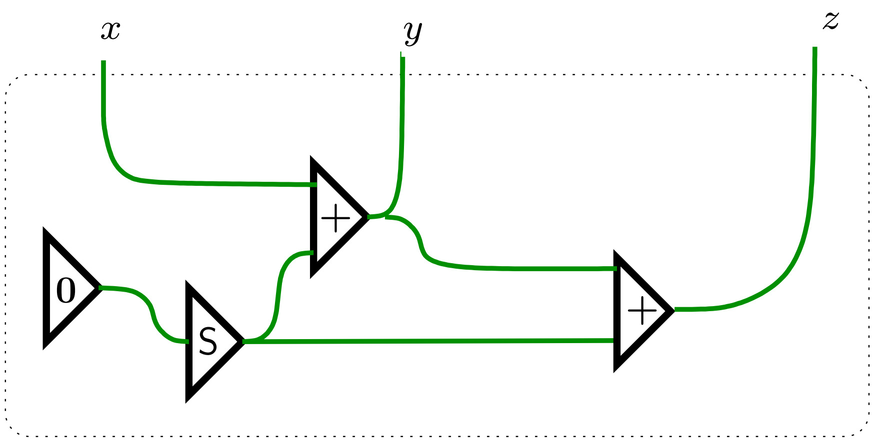

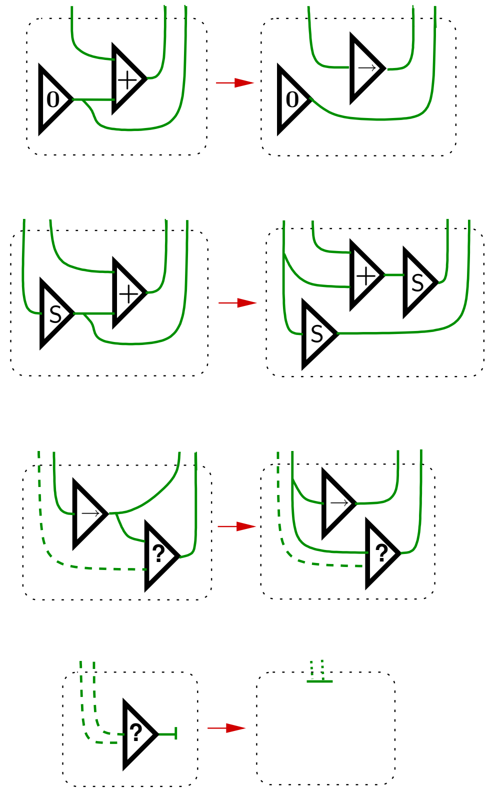

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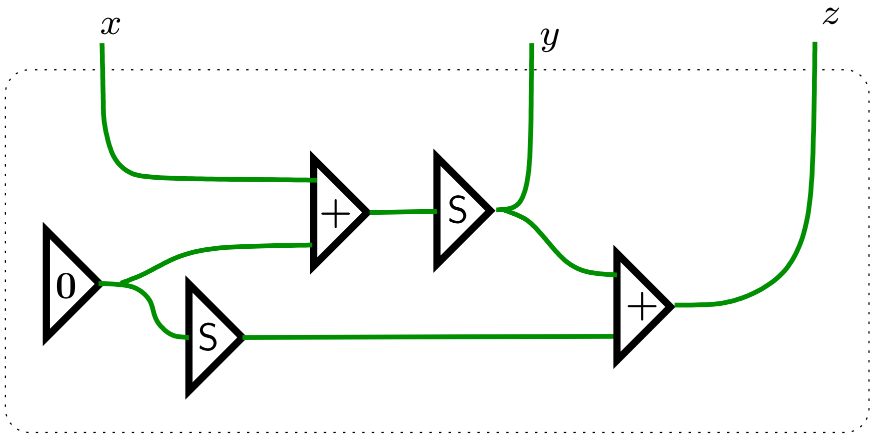

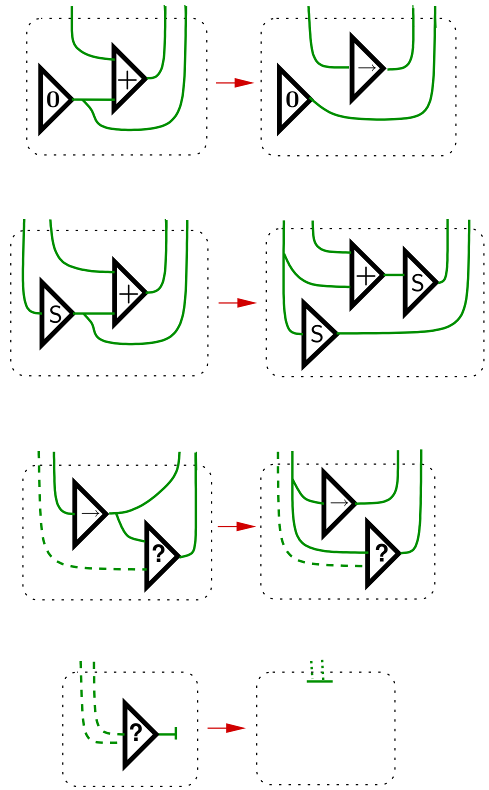

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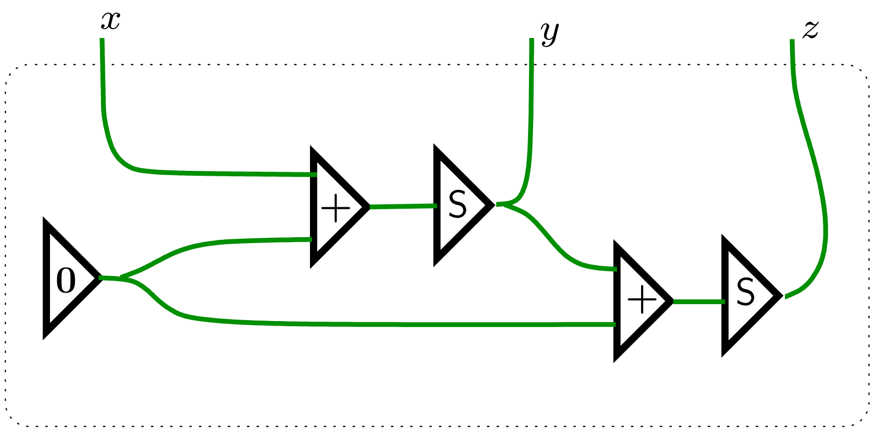

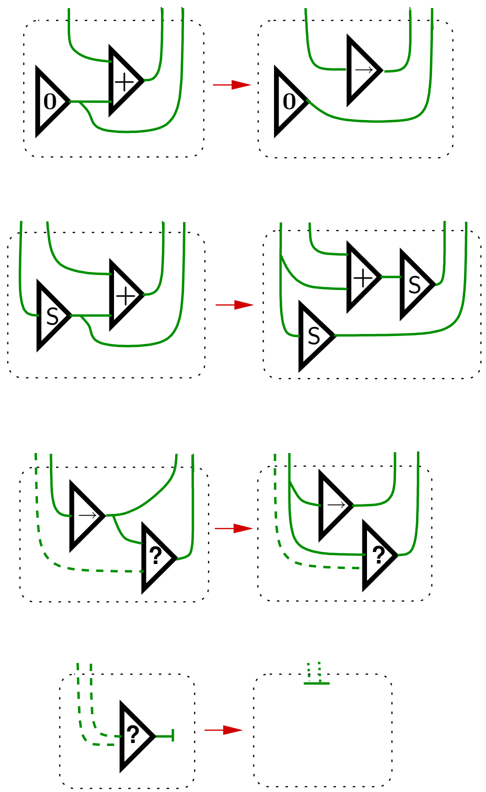

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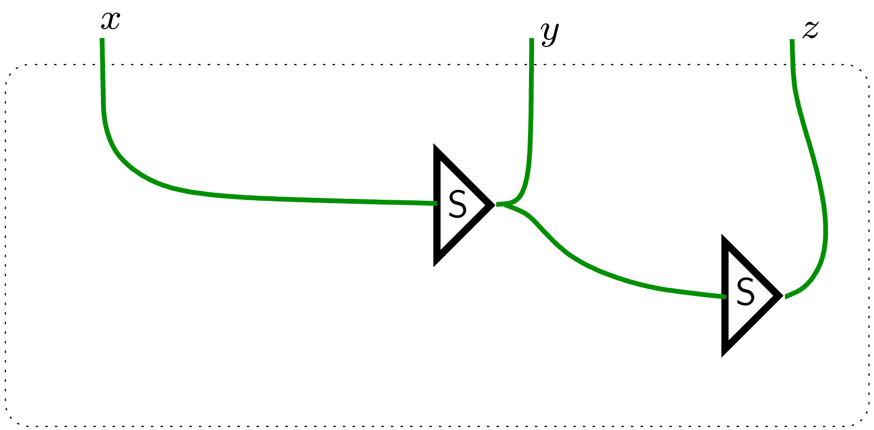

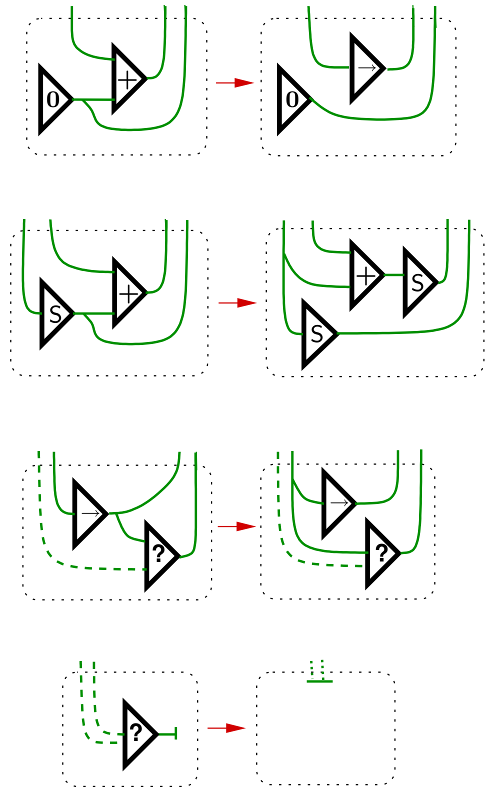

MANY-ONE SORTING

To distinguish well-formed link graphs, we often impose a **sorting** discipline on points and links. The signature $\mathcal{K}$ determines the sort of every port.

In **many-one sorting** there are two sorts $\{\mathtt{s}, \mathtt{t}\}$; source and target. A link may contain many number of $\mathtt{t}$-points, and:

- a *closed* link has exactly one $\mathtt{s}$-point;
- an *open* $\mathtt{s}$-link has exactly one $\mathtt{s}$-point;
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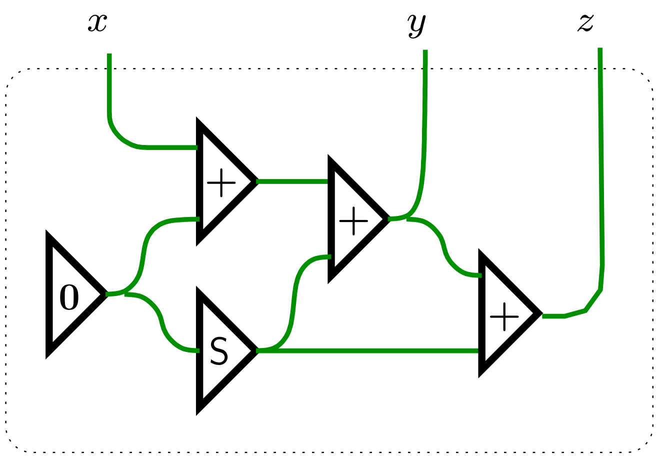
Apply this to **arithmetic nets**:



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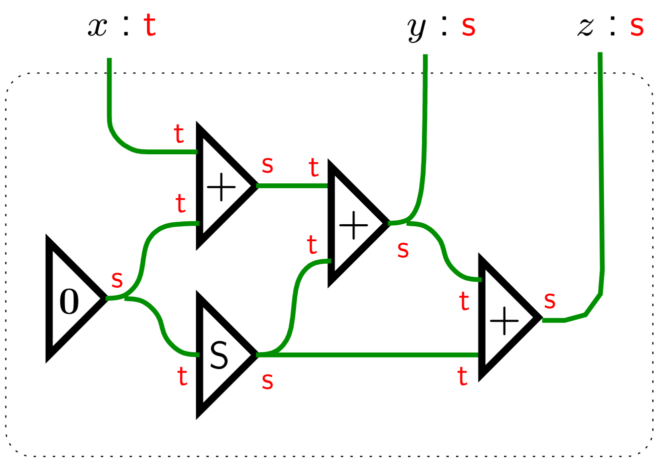
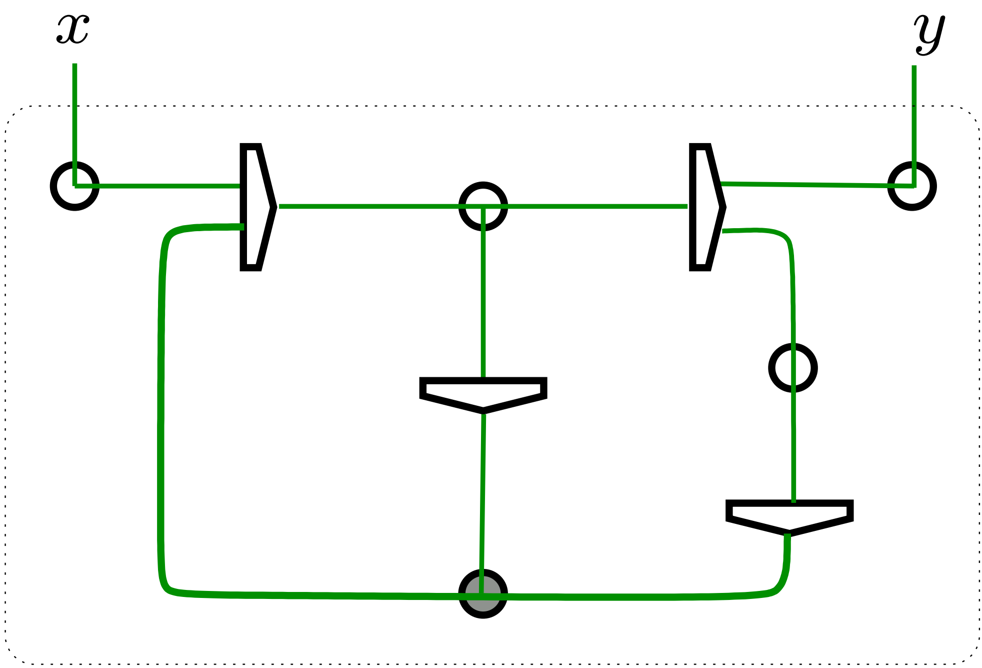


MANY-ONE SORTING for PETRI NETS

To avoid having two conditions (or two places) contiguous, all **transition** ports have sort t and all **condition** ports have sort s.



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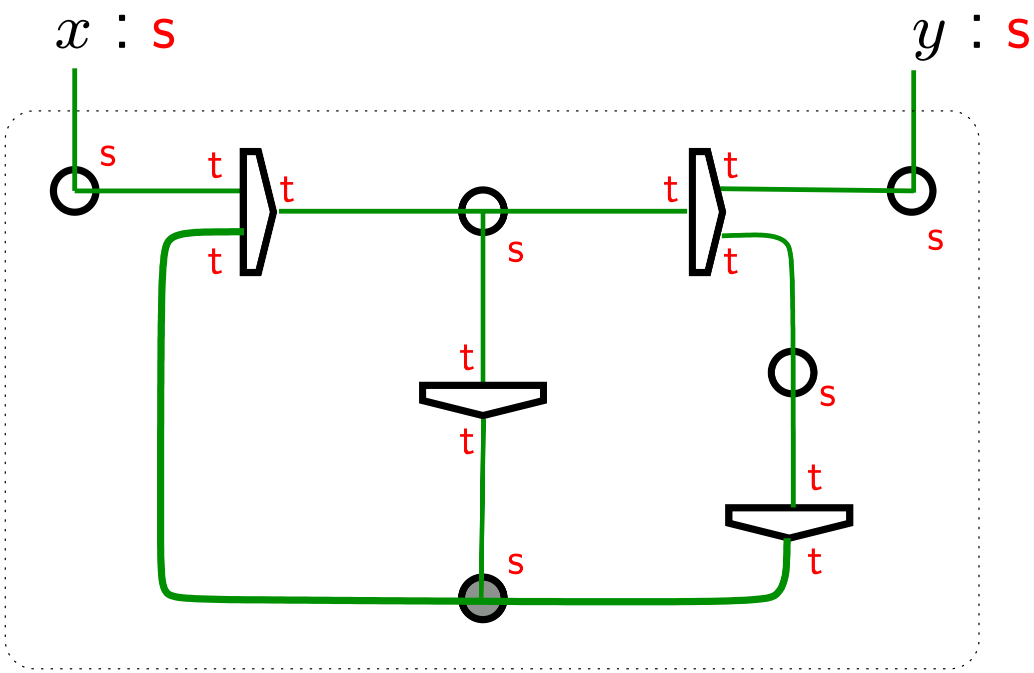


‘RAW’ LABELLED TRANSITION SYSTEM **Á($\mathcal{L}$)**

Equip **Á** with a **labelled transition system** $\mathcal{L} = (\mathsf{Ag}, \mathsf{Lab}, \mathsf{Trans})$ having **agents** Ag (ground arrows), **labels** Lab, and **transitions** Trans of form

$$a \xrightarrow{\ell} a'$$

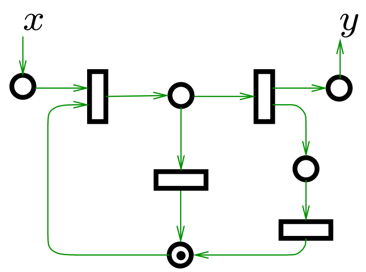
where $a, a' \in \mathsf{Ag}$ and $\ell \in \mathsf{Lab}$.

In a transition, think of ℓ as **observing** the agent a. This leads to equivalences such as **bisimilarity**, the largest symmetric relation $\sim$ such that

if $a \sim b$ *and* $b \xrightarrow{\ell} b'$, *then* $a \xrightarrow{\ell} a'$ *for some* $a' \sim b'$.

'RAW' TRANSITIONS FOR CONDITION-EVENT NETS

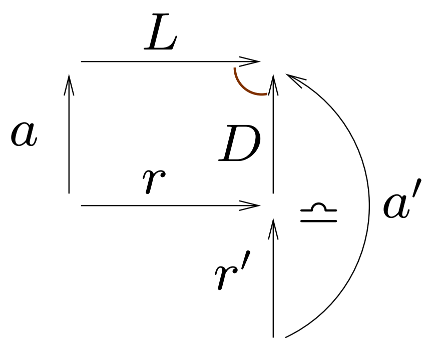
As **agents** Ag, choose c-e nets with named conditions:



As **labels** Lab , choose
$$\begin{cases} +x & \text{mark the condition named } x \\ -x & \text{unmark the condition named } x \\ \tau & \text{perform a reaction internally .} \end{cases}$$

This yields **'raw' bisimilarity,** $\simeq$. Is it a **congruence**? How does it compare with standard bisimilarity derived in the theory of link graphs?

STANDARD TRANSITIONS AND BISIMILARITY



Standard contextual LTS $\mathsf{ST} = (\mathsf{Ag}_{\mathsf{ST}}, \mathsf{Trans}_{\mathsf{ST}})$:

- $\mathsf{Ag}_{\mathsf{ST}}$ has all ground arrows
- $\mathsf{Trans}_{\mathsf{ST}}$ has all **minimal** transitions.

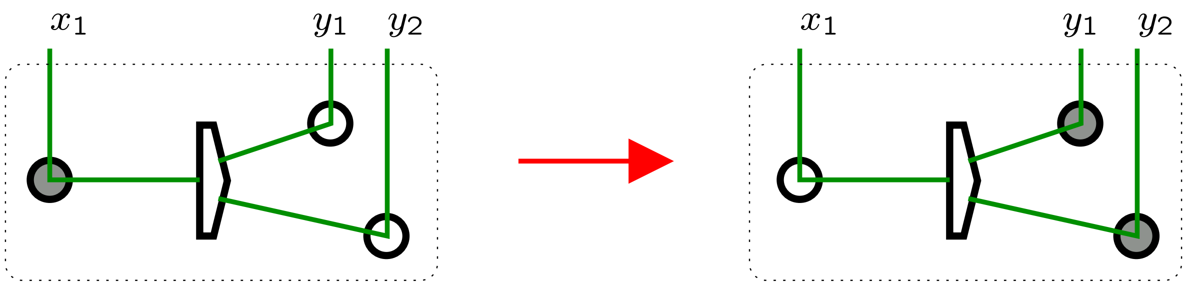
Standard bisimilarity: the largest symmetric relation $\sim$ such that

if $a \sim b$ *and* $b \xrightarrow{L} b'$, *with* La *defined, then* $a \xrightarrow{L} a'$ *for some* $a' \sim b'$.

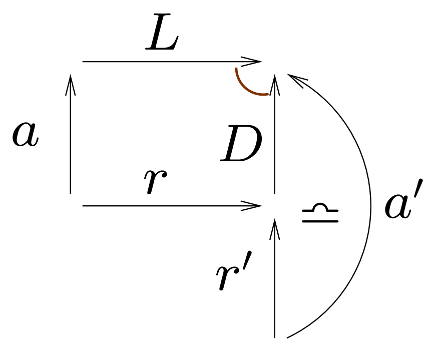
THEOREM: In any reactive system with RPOs, standard bisimilarity $\sim$ is a congruence. (*Leifer, Milner*)

DERIVED TRANSITIONS for PETRI NETS

A typical reaction rule is **simple**:



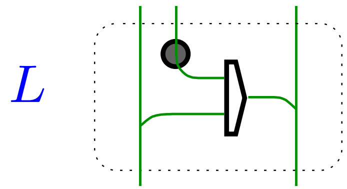
THEOREM: In a **simple** reactive system, for bisimilarity one need check only the **engaged** transitions: those where a and r share at least a node.



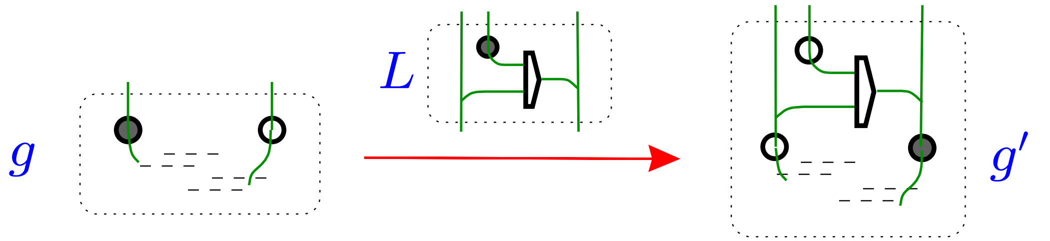
What do these transitions look like?

ENGAGED TRANSITIONS and BISIMILARITY for PETRI NETS

A typical **label** L has a single event node; it expects the agent g to supply some pre-and post-conditions:



A typical L-transition:



These **derived** labels differ from the **raw** labels $+x, -x, \tau$. But:

THEOREM : standard bisimilarity $\sim$ and raw bisimilarity $\simeq$ coincide.

— and it follows that both are **congruences**.

CONCLUSION

Bigraphs explore:

- **the modelling power of two or more orthogonal structures**
- **how to model virtual space just like physical space**
- **how to coordinate different reactive systems**

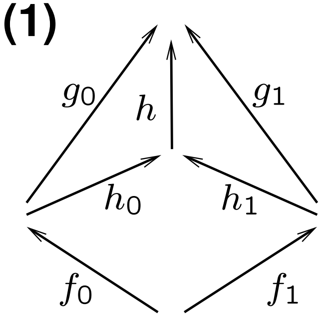
For more detail, find links at
`http://www.cl.cam.ac.uk/users/rm135/`.

RPOS AND IPOS IN AN S-CATEGORY

Write $\vec{f}$ for f_0, f_1.

Call $\vec{g}$ a **bound** for $\vec{f}$ if $g_0 f_0 = g_1 f_1$.

(1)



(1) A **relative bound** $(\vec{h}, h)$ **for** $\vec{f}$ **to** $\vec{g}$**.**

RPOS AND IPOS IN AN S-CATEGORY

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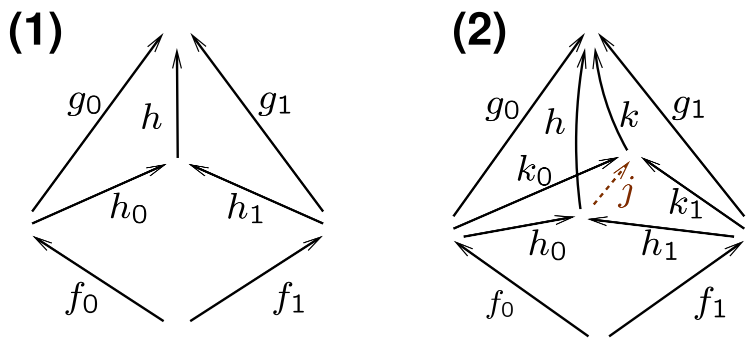


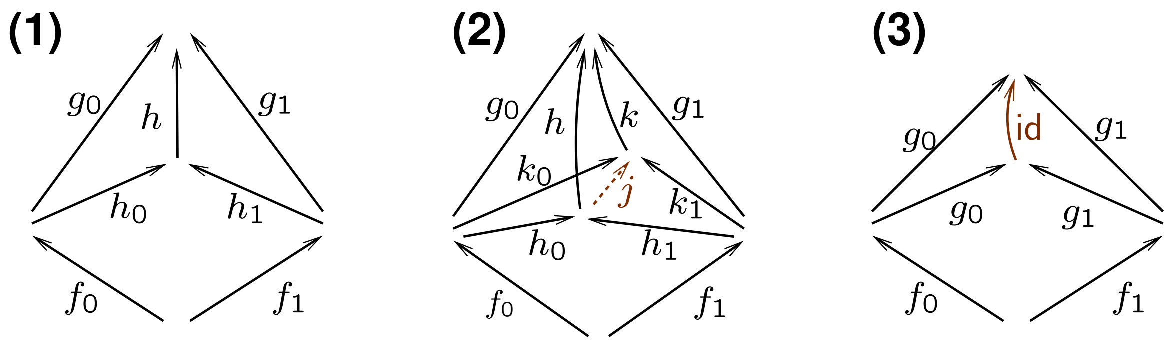
(1) A **relative bound** $(\vec{h}, h)$ **for** $\vec{f}$ **to** $\vec{g}$.

(2) A **relative pushout (RPO)** $(\vec{h}, h)$ **for** $\vec{f}$ **to** $\vec{g}$ **:** For any other relative bound $(\vec{k}, k)$, there is a unique mediator j.

RPOS AND IPOS IN AN S-CATEGORY

Write $\vec{f}$ for f_0, f_1.

Call $\vec{g}$ a **bound** for $\vec{f}$ if $g_0 f_0 = g_1 f_1$.



(1) A **relative bound** $(\vec{h}, h)$ **for** $\vec{f}$ **to** $\vec{g}$.

(2) A **relative pushout (RPO)** $(\vec{h}, h)$ **for** $\vec{f}$ **to** $\vec{g}$ **:** For any other relative bound $(\vec{k}, k)$, there is a unique mediator j.

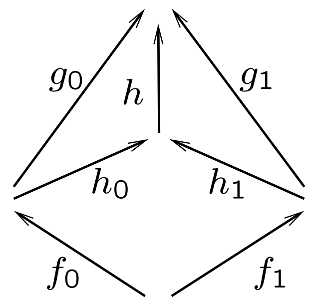
(3) An **idem pushout (IPO)** $\vec{g}$ **for** $\vec{f}$ **:** $(\vec{g}, \text{id})$ is an RPO for $\vec{f}$ to $\vec{g}$.

PROPERTIES OF RPOS and IPOS

(1) An RPO for $\vec{f}$ to $\vec{g}$ is unique up to isomorphism.

(2) If $(\vec{h}, h)$ is an RPO for $\vec{f}$, then $\vec{h}$ is an IPO for $\vec{f}$.

(3) If $\vec{h}$ is an IPO for $\vec{f}$, and an RPO exists for $\vec{f}$ to $\vec{g}$, then $(\vec{h}, h)$ is such an RPO.



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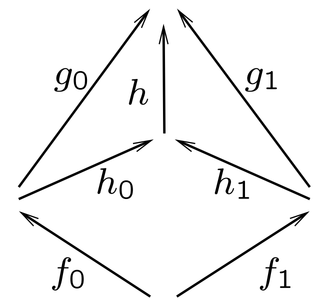

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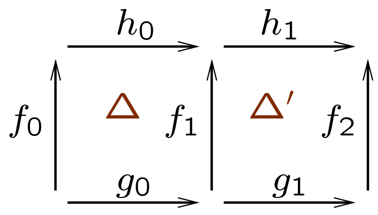
(4)
- If $\triangle$ and $\triangle'$ are IPOs, so is the whole.
- If the whole and $\triangle$ are IPOs, so is $\triangle'$.

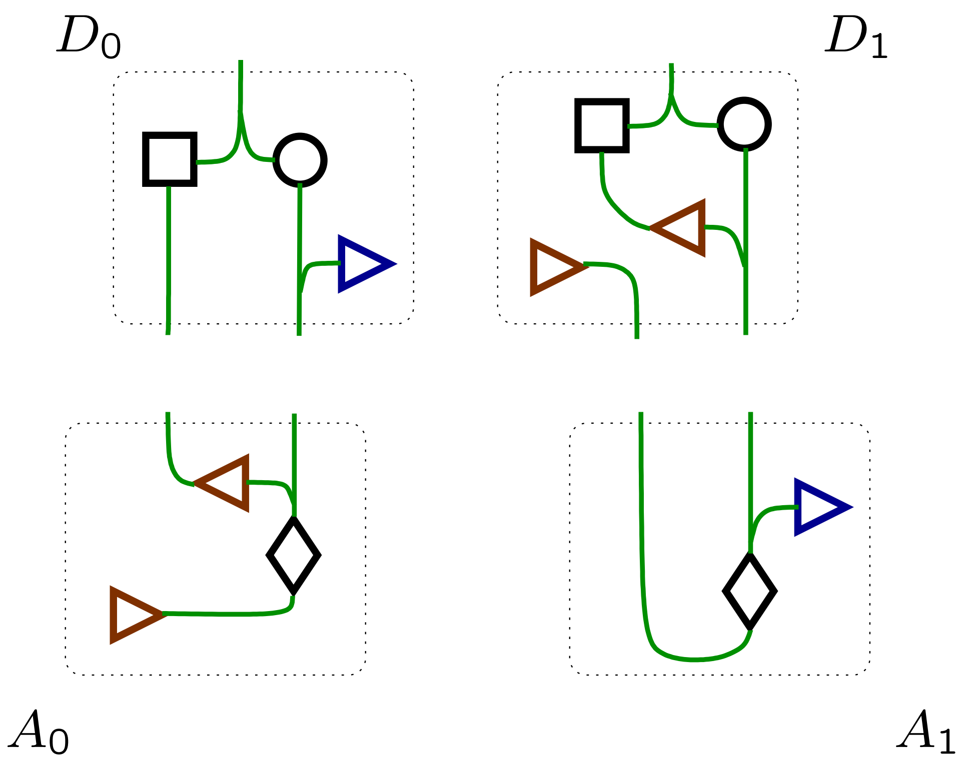
(when enough RPOs exist)



(5) In an s-category, if $(\vec{f}, \vec{h}) \doteq (\vec{f}', \vec{h}')$ and one is an IPO, so is the other.

AN RPO FOR LINK GRAPHS (1)

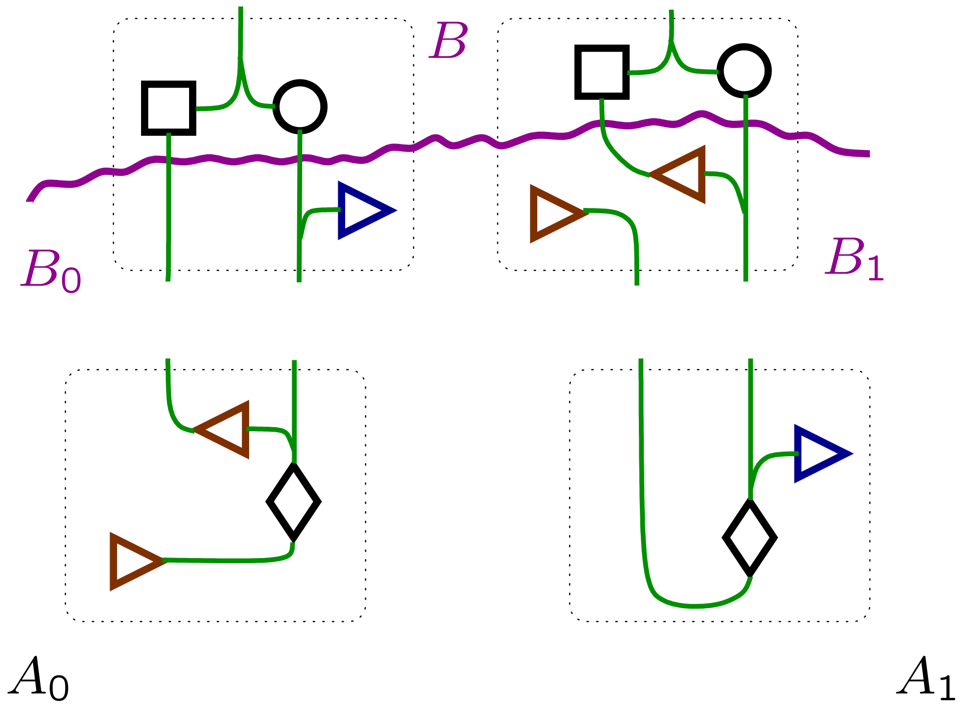
(D_0, D_1) is a bound for (A_0, A_1)



AN RPO FOR LINK GRAPHS (2)

(D_0, D_1) is a bound for (A_0, A_1)

Truncate D_0, D_1 to make an RPO (B_0, B_1, B)

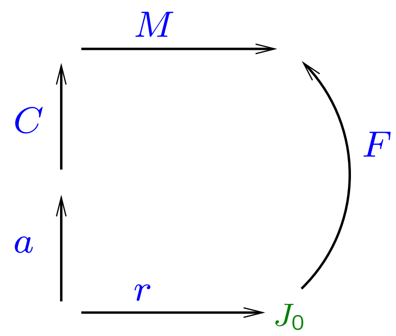


THEOREM: Link graphs have RPOs

PROVING CONGRUENCE of BISIMILARITY

To prove that $a \sim b \Rightarrow Ca \sim Cb$, it is sufficient to show that $\mathcal{S} = \{(Ca,\, Cb) \mid a \sim b\}$ is a bisimulation up to $\simeq$.

(1) Let $a \sim b$, and suppose $\boxed{Ca \xrightarrow{M} a''}$ with ground rule (r, r'). So $a'' \simeq Fr'$.



We must find b'' *for which* $Cb \xrightarrow{M} b''$ *and* $a'' \mathcal{S} b''$.

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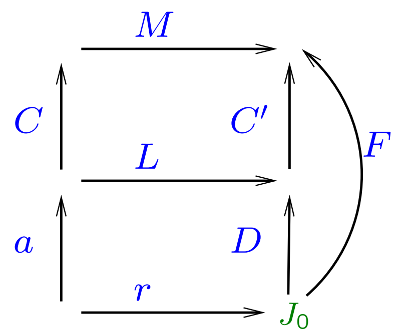
(2) Take the RPO of (a, r). Then $a \xrightarrow{L} a' \stackrel{\text{def}}{=} Dr'$. So $\boxed{a'' \simeq C'a'}$.



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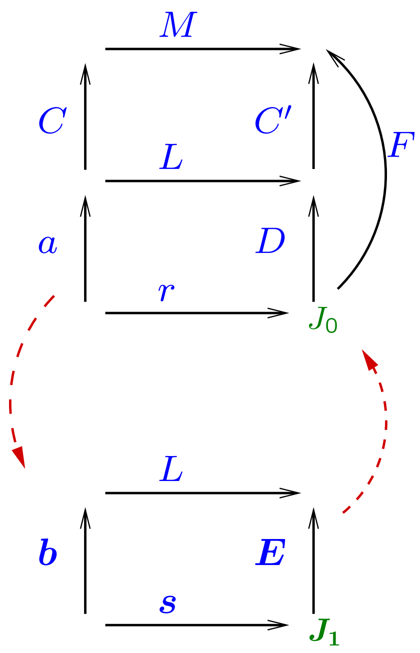
(3) From $a \sim b$ deduce $b \xrightarrow{L} b'$ with $\boxed{a' \sim b'}$.



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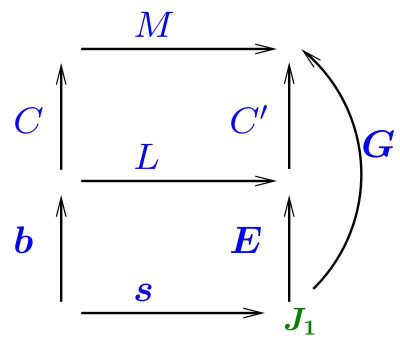

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(3) From $a \sim b$ deduce $b \xrightarrow{L} b'$ with $\boxed{a' \sim b'}$.

(4) Swap lower IPOs; deduce $\boxed{Cb \xrightarrow{M} b'' \stackrel{\text{def}}{=} C'b'}$. ■