

# **BIGRAPHS: coordinating mobile computation**

**March 2006**

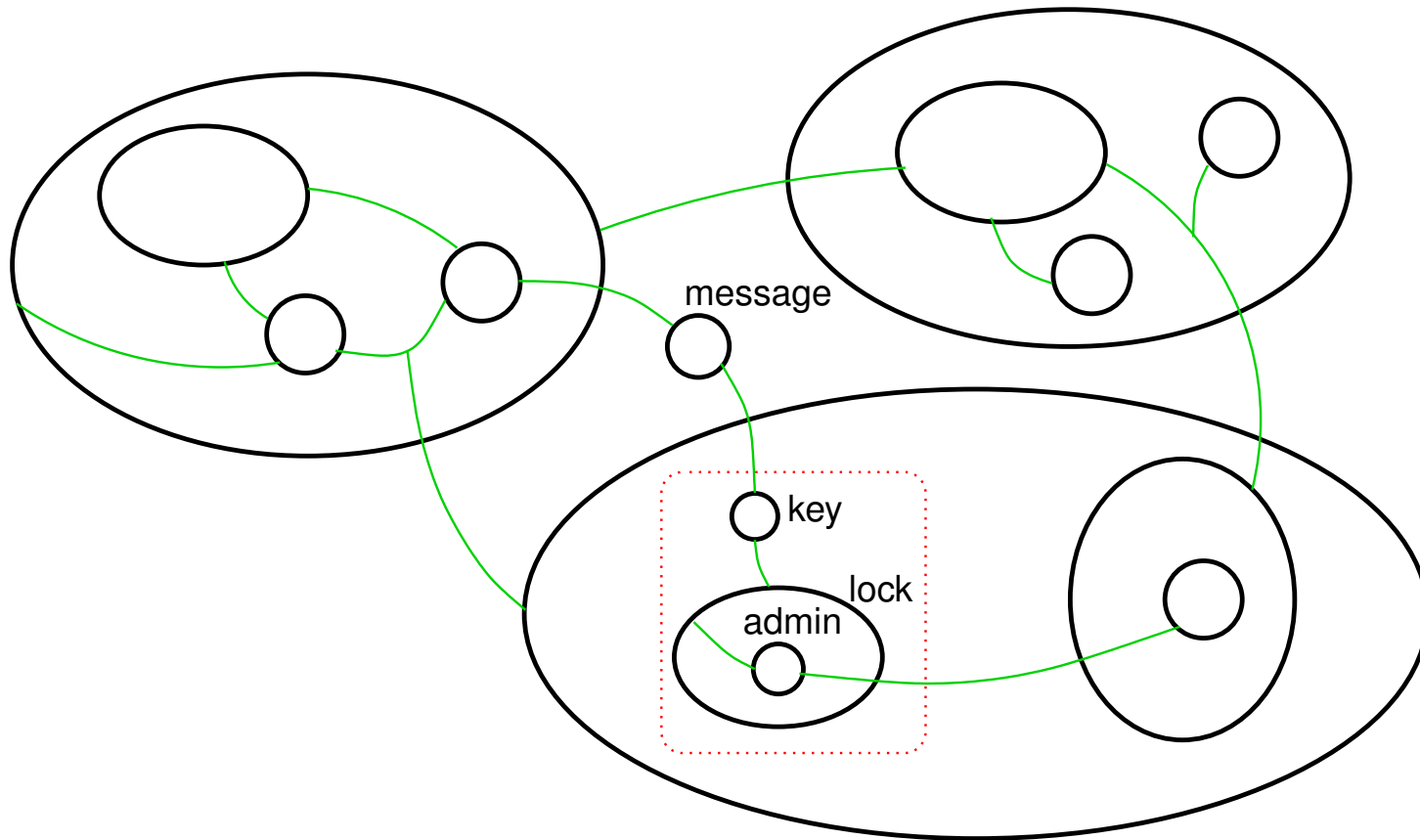
**Robin Milner**

including joint work with **Jamey Leifer** and **Ole Jensen**

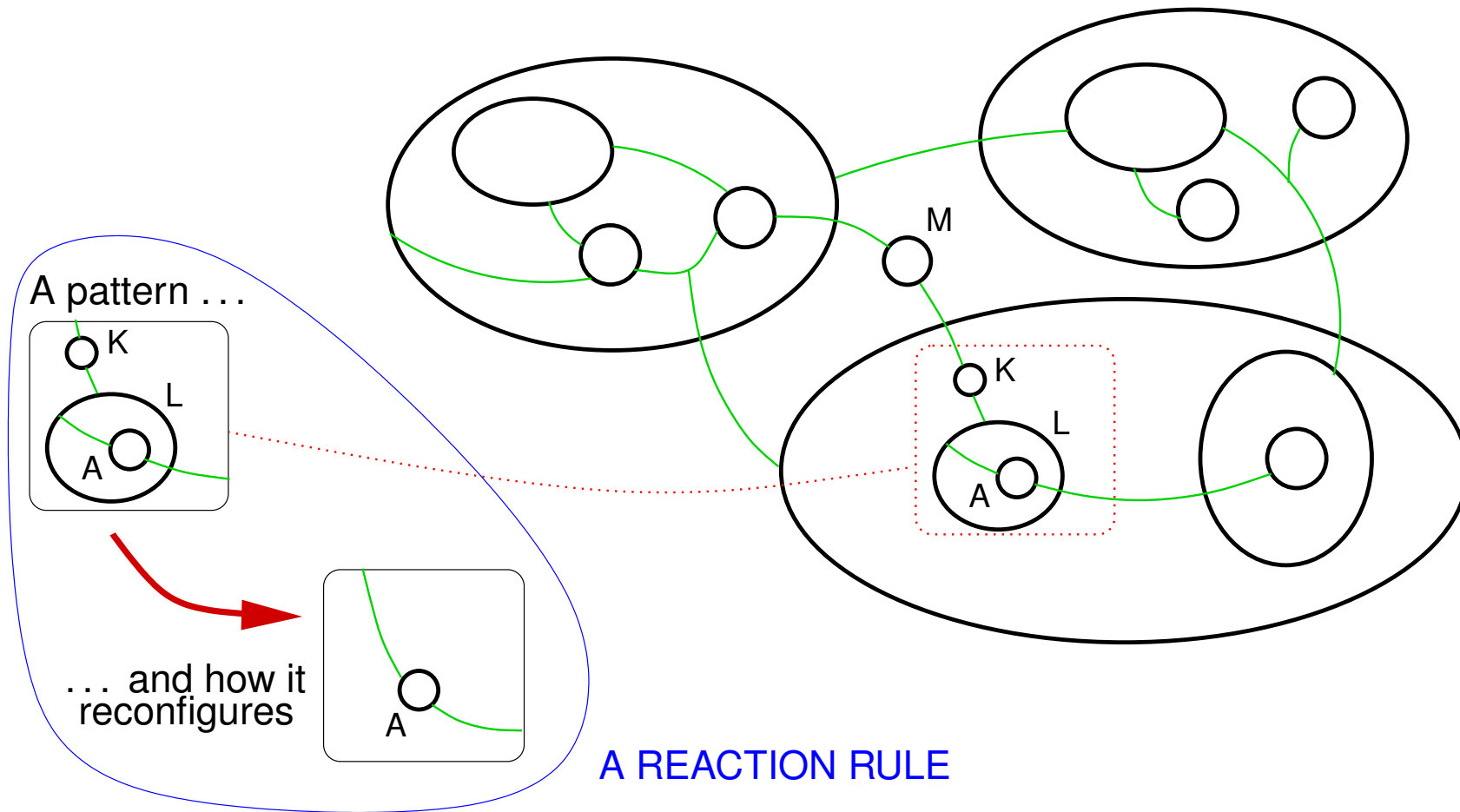
- Part I**      **Examples of bigraph reactions**
- Part II**    **S-categories, reactions and transitions**
- Part III**   **Link graphs, sorting and Petri nets**

# **Part I      EXAMPLES OF BIGRAPH REACTIONS**

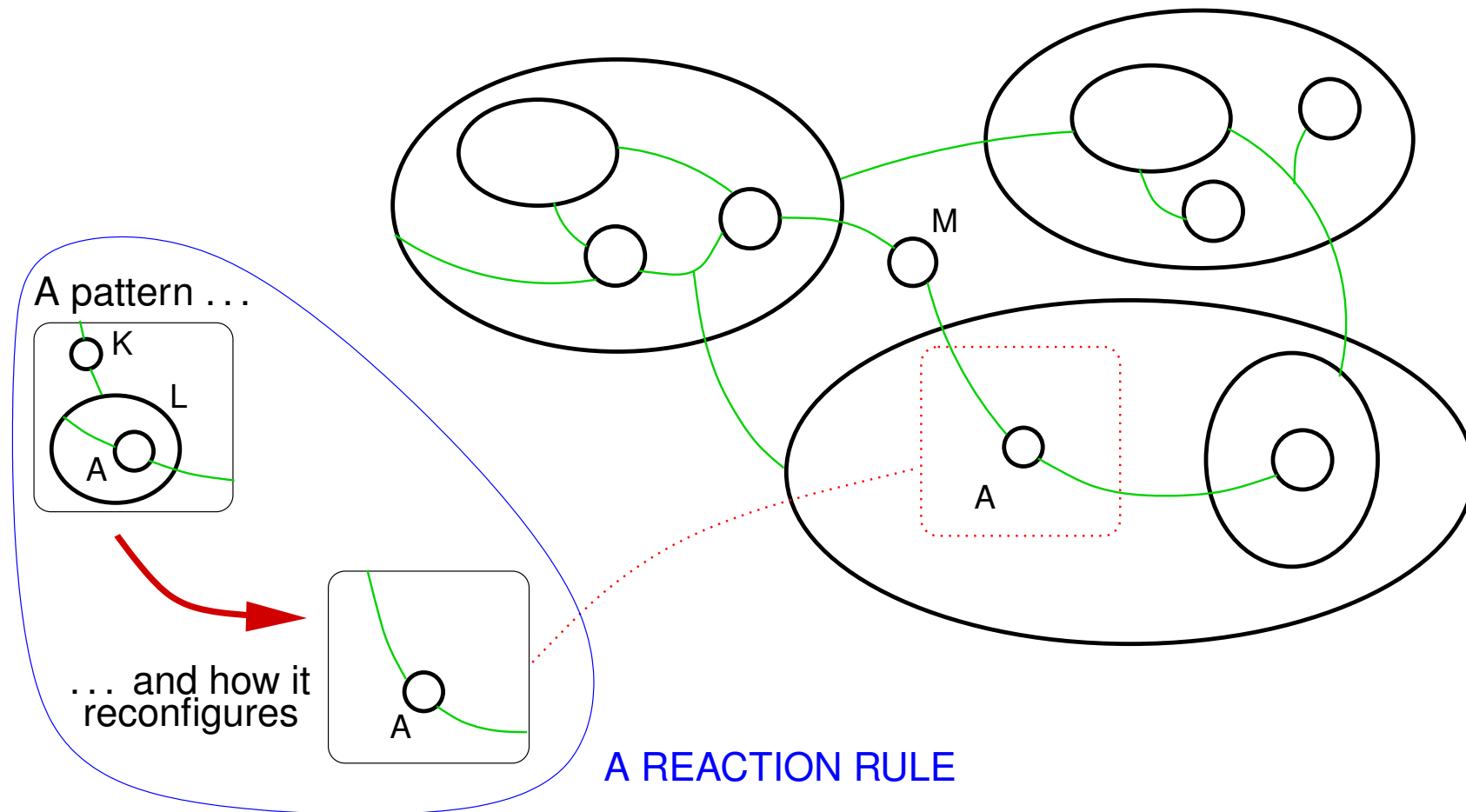
## A TYPICAL BIGRAPH



# HOW A SYSTEM MAY RECONFIGURE .....



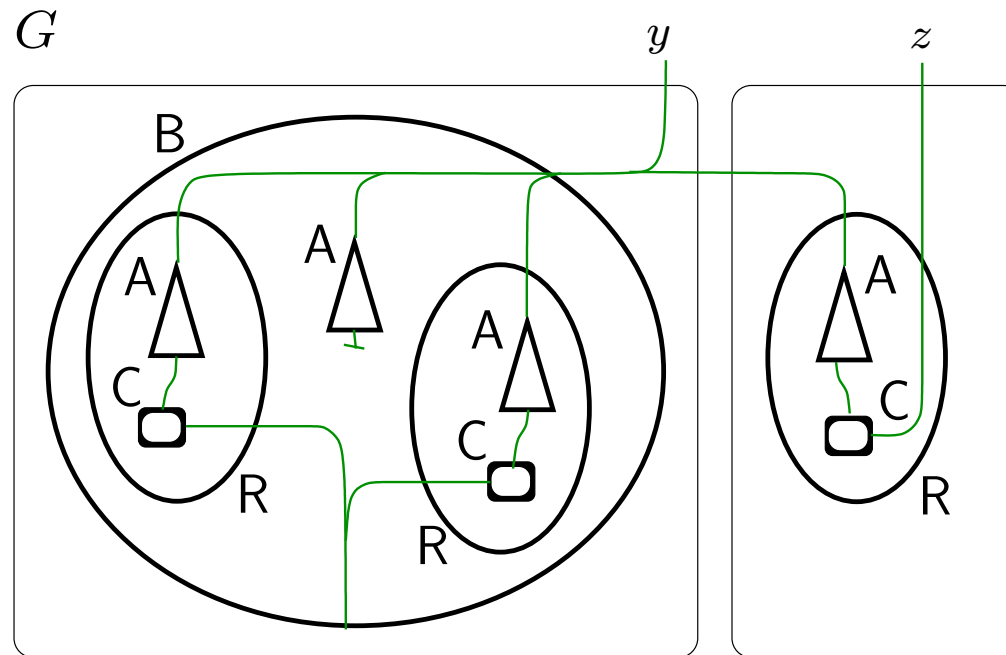
# ..... AND THE NEW CONFIGURATION



## INTERACTIONS IN A BUILT ENVIRONMENT (1)

A bigraph  $G$  with two regions, representing a conference call

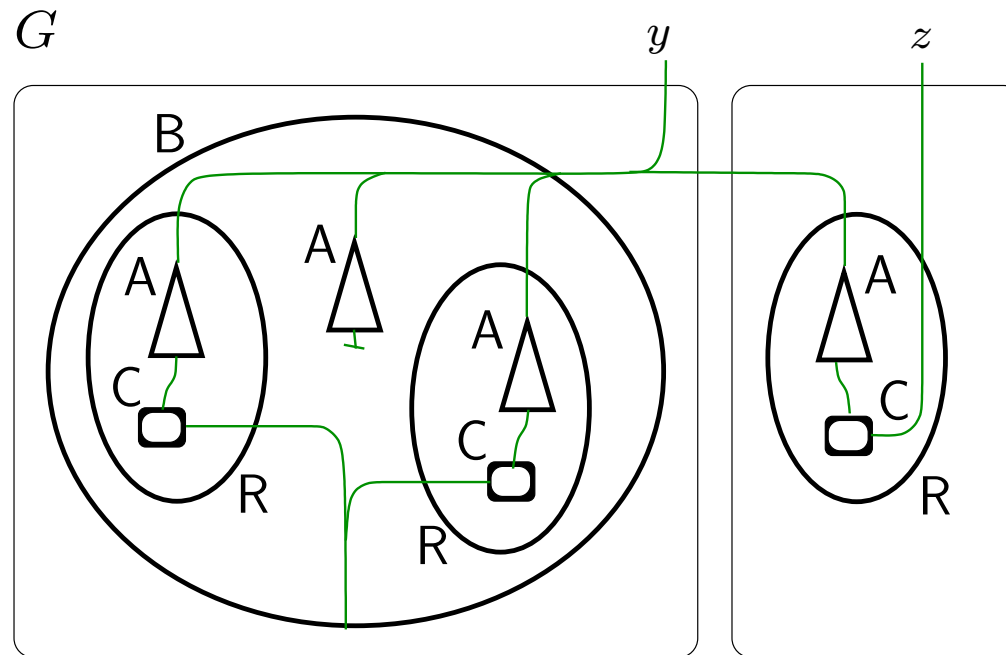
B BUILDING  
R ROOM  
A AGENT  
C COMPUTER



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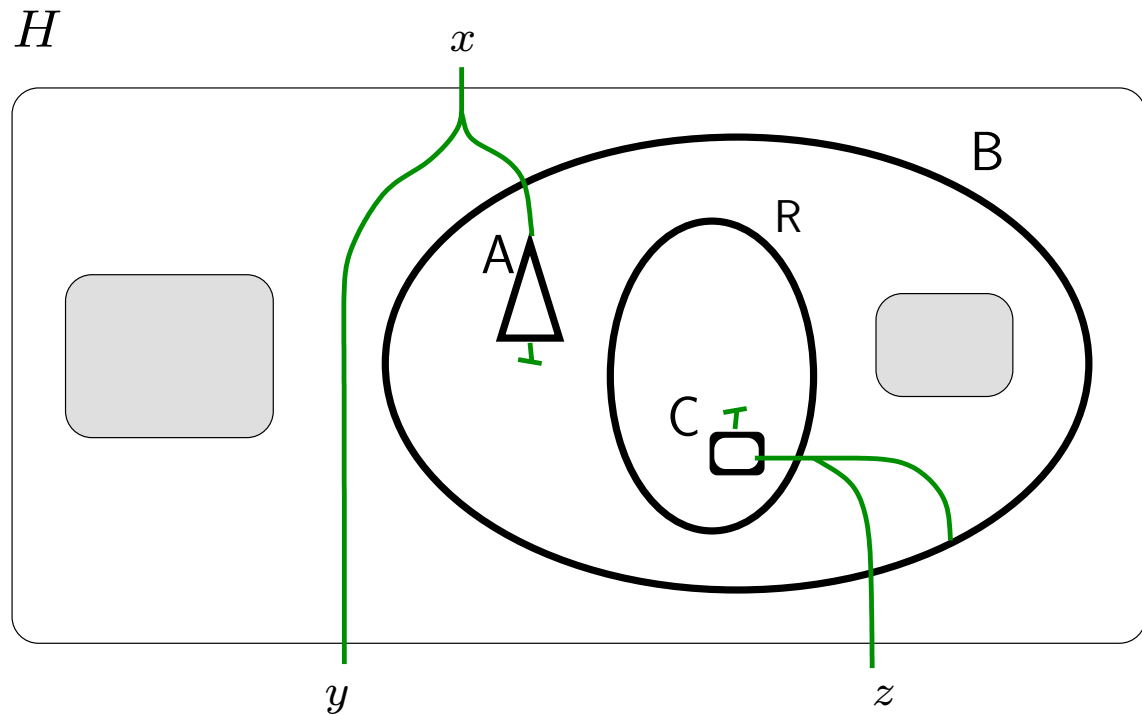
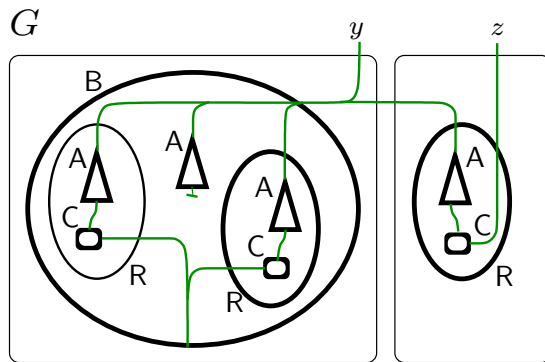


$$\begin{aligned} & /x (Bx(R(/u Ayu | Cux) | /u Ayu)R(/u Ayu | Cux)) \\ & \parallel R(/v Ayv | Cvz) \end{aligned}$$

*algebraic form:*

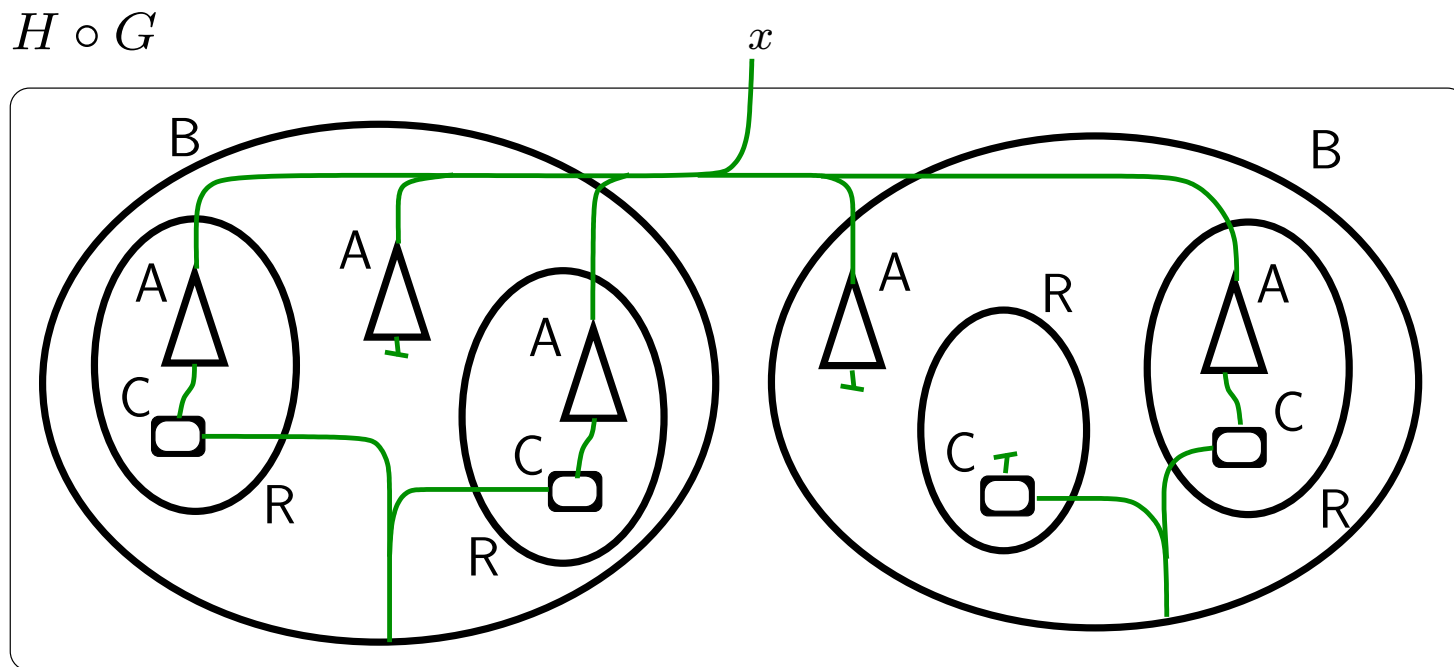
## INTERACTIONS IN A SENTIENT BUILDING (2)

A host environment  $H$ ,  
which  $G$  may inhabit



## INTERACTIONS IN A SENTIENT BUILDING (3)

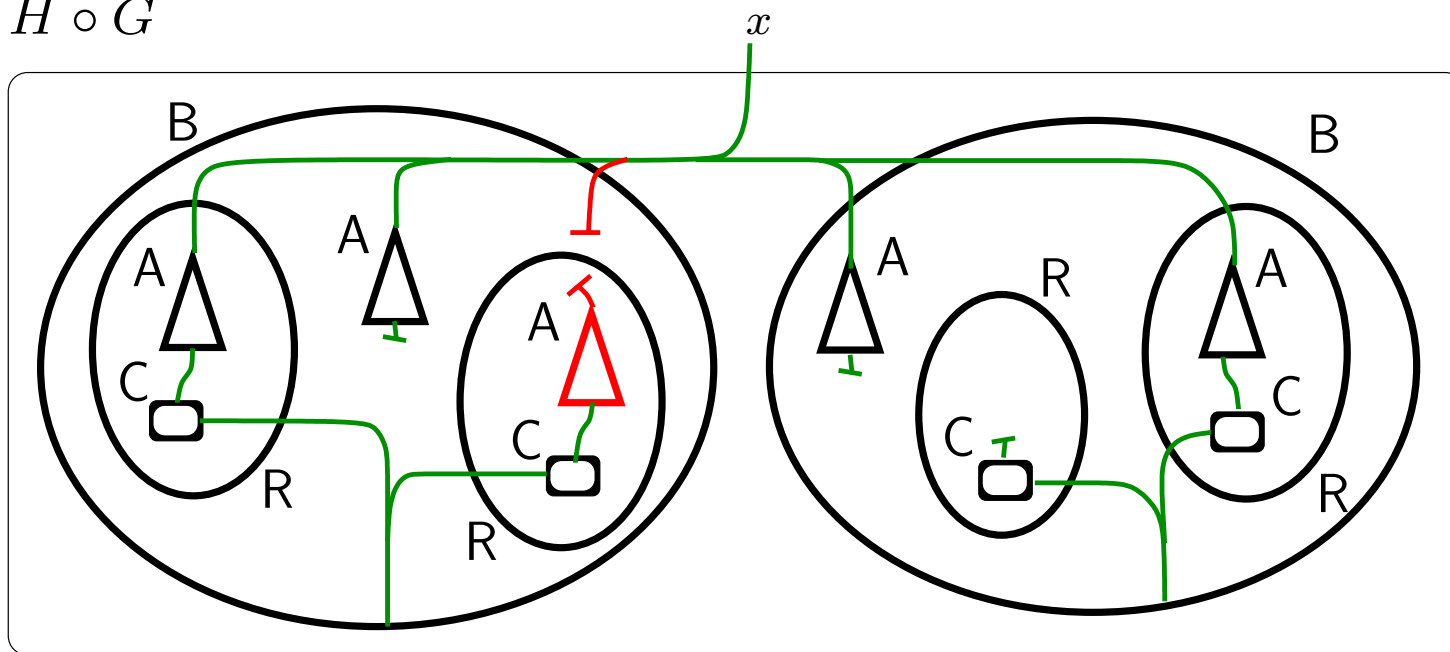
The larger environment,  $H \circ G$ .



## INTERACTIONS IN A SENTIENT BUILDING (4)

One agent leaves the call!

$H \circ G$

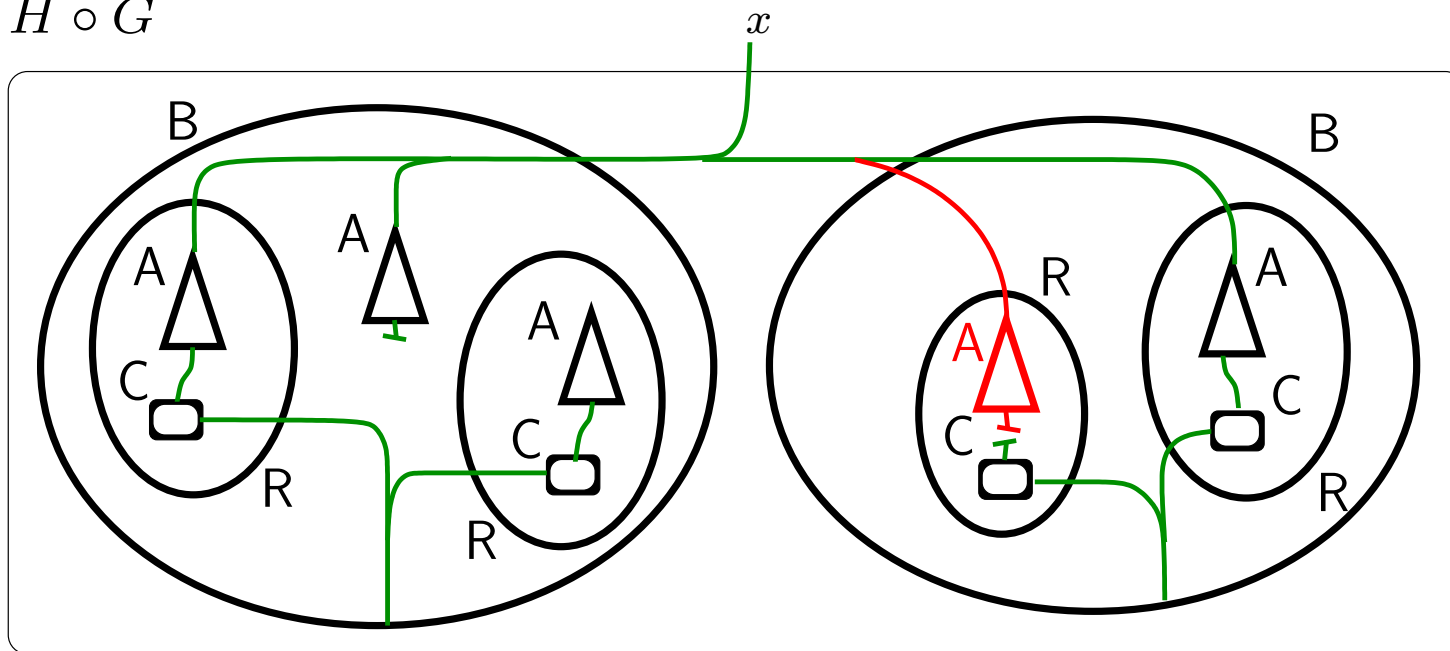


## INTERACTIONS IN A SENTIENT BUILDING (5)

One agent leaves the call

Another moves into a room ...

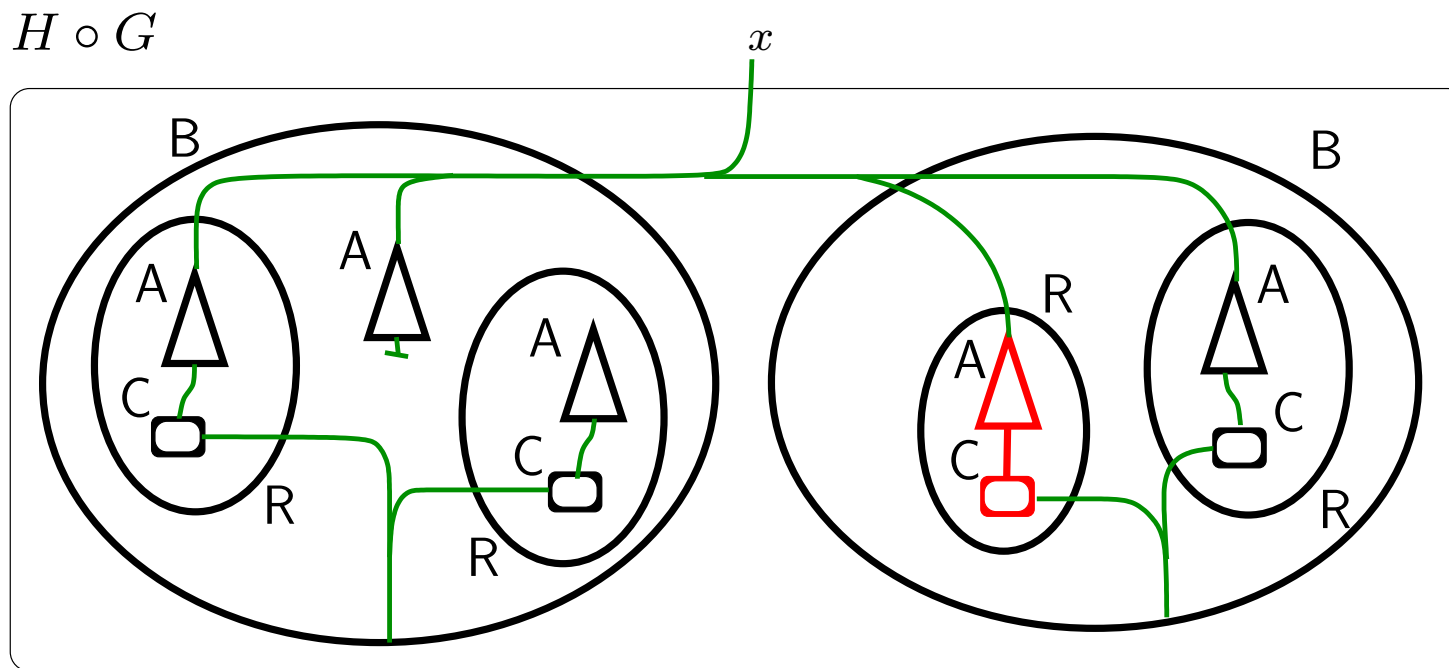
$H \circ G$



## INTERACTIONS IN A SENTIENT BUILDING (6)

The larger environment,  $H \circ G$ .

One agent leaves the call  
Another moves into a room ...  
... and is sensed and logged in!



## SIGNATURES AND CONTROLS

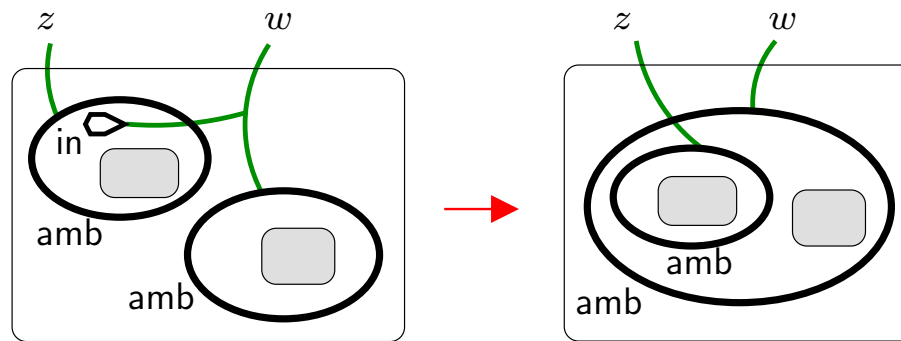
A **signature**  $\mathcal{K}$  determines a set of **controls**  $K, L, \dots$   
In a bigraph over  $\mathcal{K}$ , each node is assigned a control.

$\mathcal{K}$  also determines, for each control  $K$ :

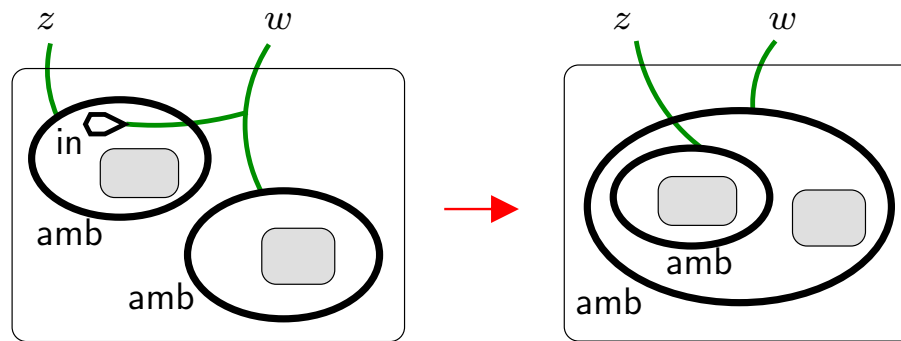
- an **arity**  $n$ : the number of **ports**
- whether **atomic**; if not, whether **active** or **passive**

**Atomic** nodes are empty. **Active** nodes permit reaction inside.

## A REACTION RULE FOR THE AMBIENT CALCULUS



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$$\text{amb}_z.(\text{in}_w \mid d) \mid \text{amb}_w.e \longrightarrow \text{amb}_w.(\text{amb}_z.d \mid e)$$

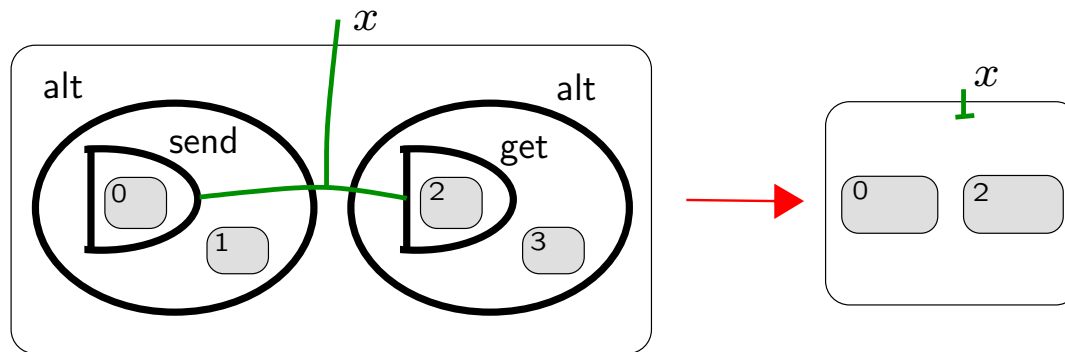
$\text{amb} : 1$  is *non-atomic* and *active*  
 $\text{in} : 1$  is *atomic*

## A REACTION RULE FOR CCS

$$(\bar{x}.P + M) | (x.Q + N) \longrightarrow P | Q$$

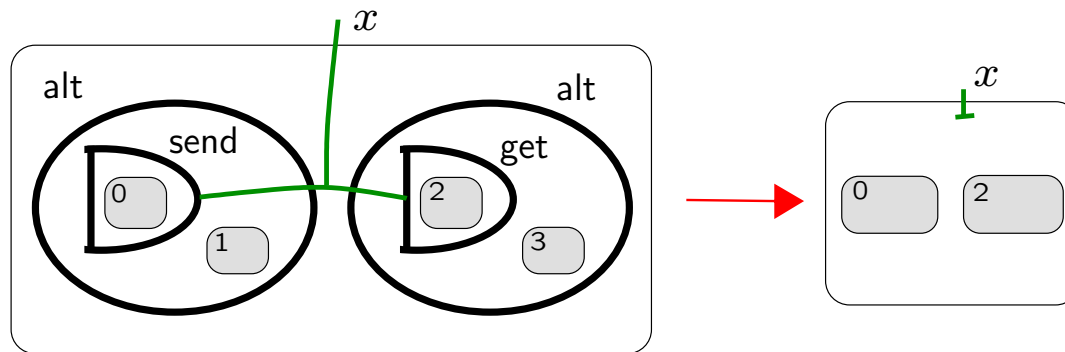
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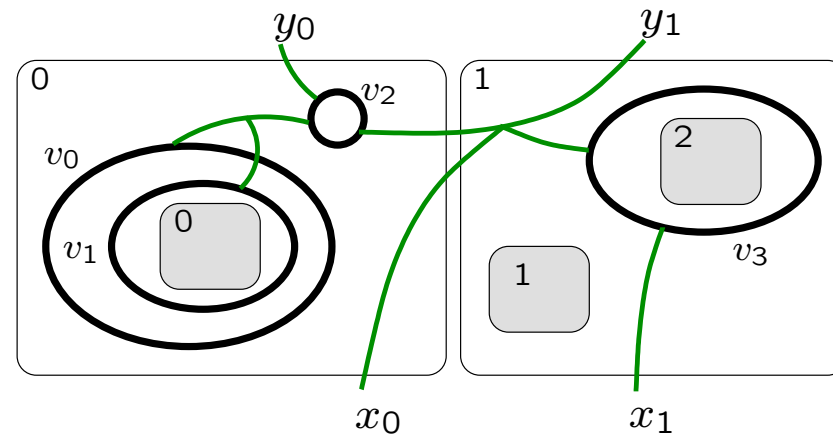


$$\text{alt.} (\text{send}_{x.p} | m) | \text{alt.} (\text{get}_{x.q} | n) \longrightarrow x | p | q$$

*alt*, *send*, *get* are *non-atomic* and *passive*

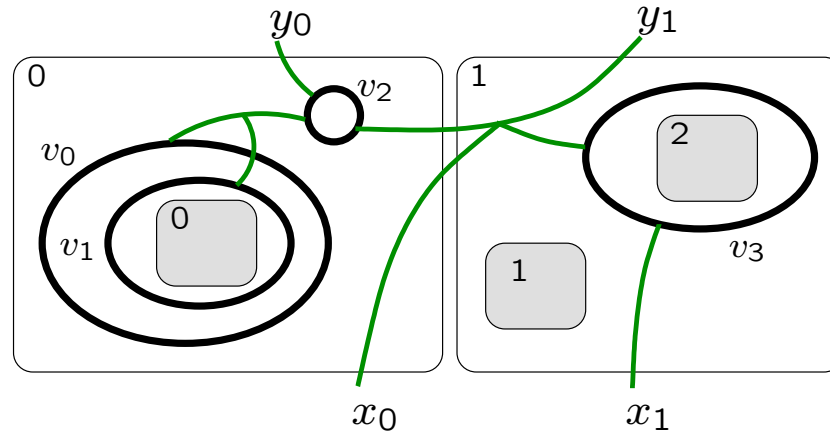
## RESOLVING A BIGRAPH INTO PARTS

**bigraph**  
 $G: \langle m, X \rangle \rightarrow \langle n, Y \rangle$

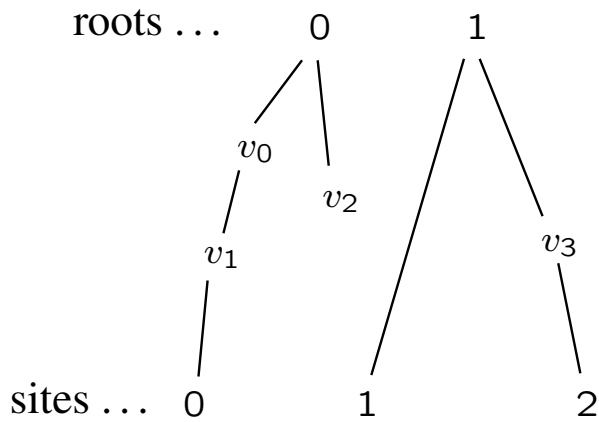


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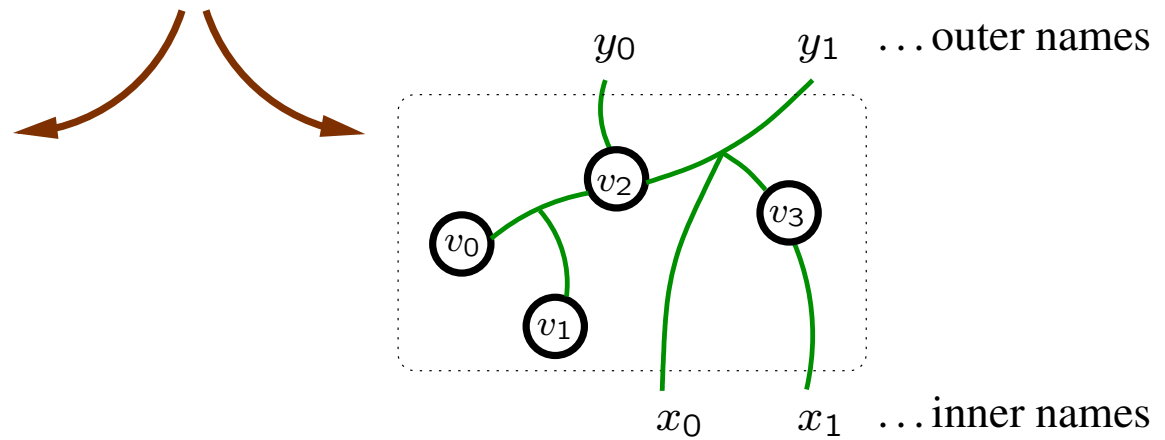
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**place graph**  
 $G^P: m \rightarrow n$

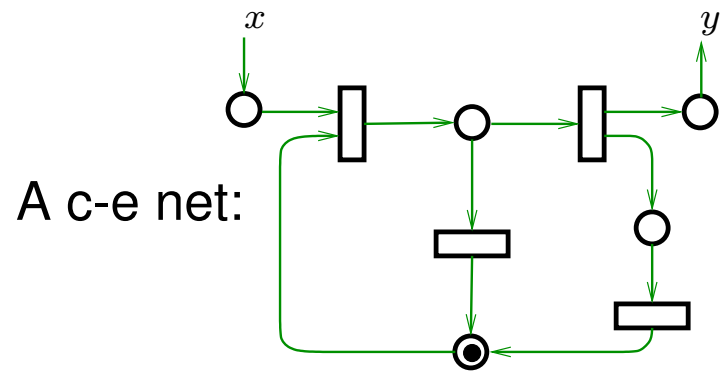


**link graph**  
 $G^L: X \rightarrow Y$



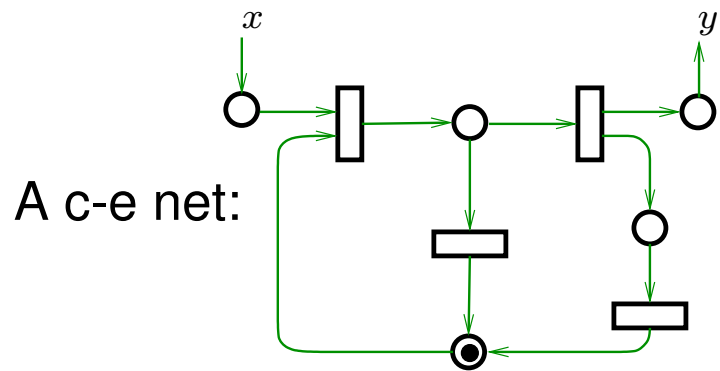
## PETRI NETS AS LINK GRAPHS

We consider **condition-event nets** with named conditions

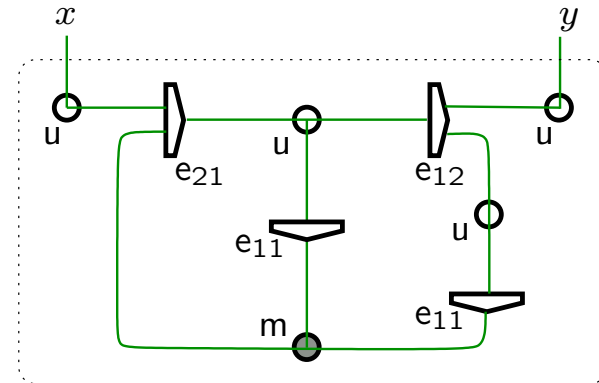


## PETRI NETS AS LINK GRAPHS

We consider **condition-event nets** with named conditions



its link graph:



$m: 1$

marked condition

$u: 1$

unmarked condition

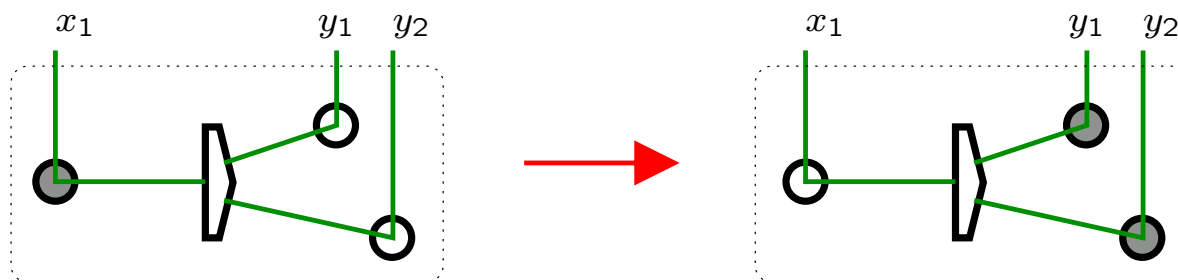
$e_{hk}: h + k$

event with  $h$  pre-conditions  
 $k$  post-conditions

## REACTION RULES FOR PETRI NETS

An event can  $e_{hk}$  'fire' iff its preconditions are all marked and its postconditions all unmarked

The rule for  $h = 1, k = 2$ :



**Part II**      **S-CATEGORIES, REACTIONS and  
TRANSITIONS**

## S-CATEGORIES

An **s-category**  $\mathcal{A}$  :

- is like a **category**: it has *objects*  $I, J, \dots$  (here called *interfaces*), and arrows  $A : I \rightarrow J$  between them. Under certain conditions we can *compose*  $A : I \rightarrow J$  and  $B : J \rightarrow K$  to form  $B \circ A : I \rightarrow K$ .

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- is **symmetric monoidal**: given suitably disjoint interfaces  $I, J$  we can form  $I \otimes J$ , their *tensor product*.  
Given  $A : I \rightarrow I'$  and  $B : J \rightarrow J'$  we can form  $A \otimes B : I \otimes J \rightarrow I' \otimes J'$ , their *tensor product*; it is often merely *juxtaposition*.

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Certain axioms hold for  $\circ$  and  $\otimes$ . For this talk, the difference between categories and s-categories can mostly be ignored.

## REACTIVE SYSTEM $\mathbf{A}(\mathcal{R})$

The empty interface  $\epsilon$ , the unit of  $\otimes$ , is called the **origin**.

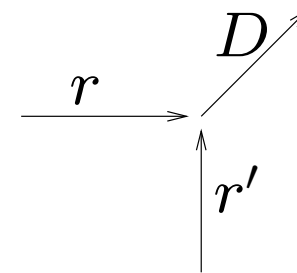
A **ground** arrow is  $a : \epsilon \rightarrow I$ , written  $a : I$ .

To form  $\mathbf{A}(\mathcal{R})$ , supply  $\mathbf{A}$  with a set  $\mathcal{R}$  of **ground reaction rules**  $(r : I, r' : I)$ .

**Reaction relation:** the smallest  $\longrightarrow$  such that

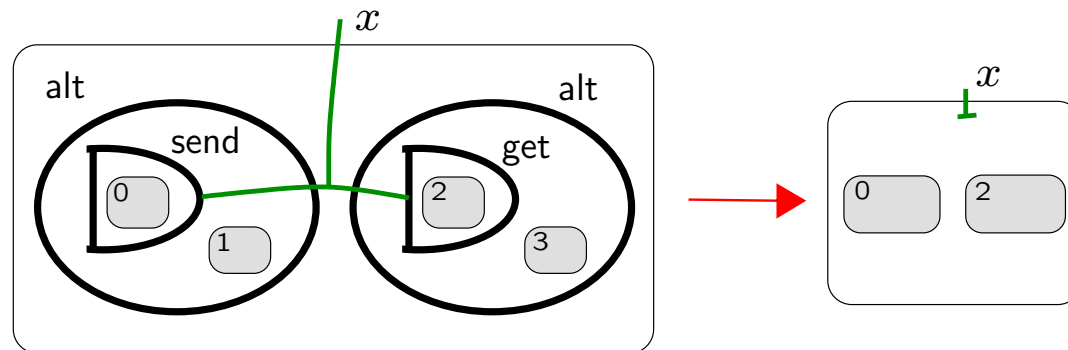
$$D \circ r \longrightarrow D \circ r'$$

for all contexts  $D$  and ground rules  $(r, r')$ .



## The REACTION RULE for CCS is **PARAMETRIC**

$$(\bar{x}.P + M) | (x.Q + N) \longrightarrow P | Q$$



**Parametric rules are defined using contexts**

## REACTION EQUIVALENCE:

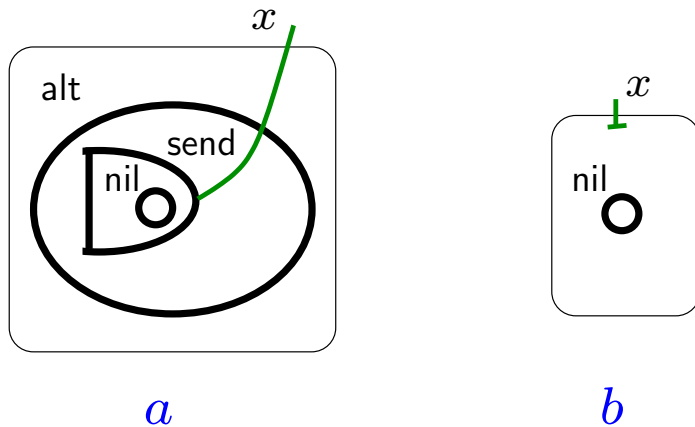
Define **reaction equivalence** of ground bigraphs as a *bisimilarity*:

if  $a \approx b$  and  $a \xrightarrow{\triangleright} a'$ , then  $b \xrightarrow{\triangleright} b'$  such that  $a' \approx b'$ .

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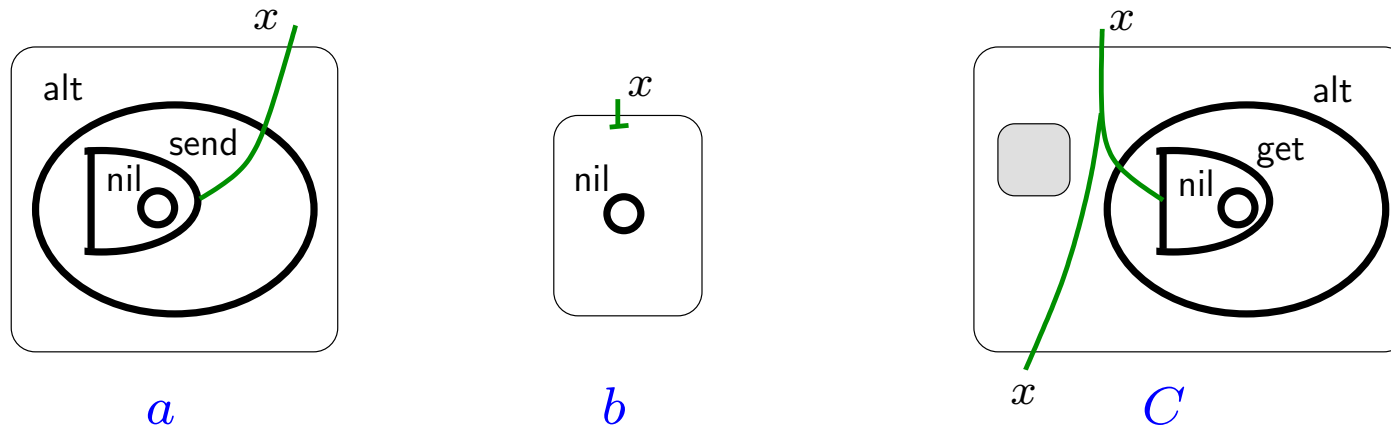


Is  $a \approx b$  here? YES!!

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Is  $a \approx b$  here? YES!! But they ‘**react with their environment**’ differently; e.g.  $C \circ a$  can react, but  $C \circ b$  cannot. So  $\approx$  is not a congruence.

**We would like a congruence!**

## 'RAW' LABELLED TRANSITION SYSTEM $\mathbf{A}(\mathcal{L})$

Equip  $\mathbf{A}$  with a **labelled transition system**  $\mathcal{L} = (\text{Ag}, \text{Lab}, \text{Trans})$  having **agents**  $\text{Ag}$  (ground arrows), **labels**  $\text{Lab}$ , and **transitions**  $\text{Trans}$  of form

$$a \xrightarrow{\ell} a'$$

where  $a, a' \in \text{Ag}$  and  $\ell \in \text{Lab}$ .

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In a transition, think of  $\ell$  as **observing** the agent  $a$ . This leads to equivalences such as **bisimilarity**, the largest symmetric relation  $\sim$  such that

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**So what kinds of LTS guarantee this kind of congruence?**

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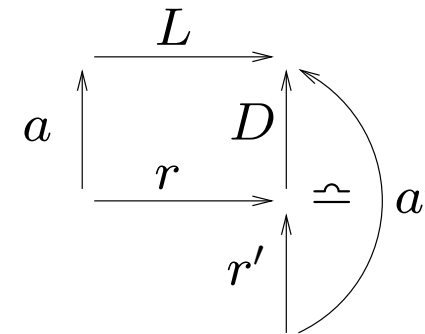
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So what kinds of LTS guarantee this kind of **congruence**?

**ANSWER: Contextual LTSs, when Relative Pushouts (RPOs) exist.**

## CONTEXTUAL TRANSITION SYSTEMS

**Contextual transition** for  $\mathbf{A}(\mathcal{R})$ : a transition  $a \xrightarrow{L} a'$  where  $L$  and  $D$  are **contexts** making the diagram commute, for some rule  $(r, r')$  in  $\mathcal{R}$ .



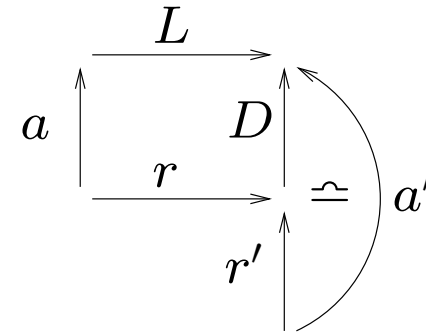
**Contextual transition system:**  $\mathcal{L} = (\text{Ag}, \text{Trans})$ , where

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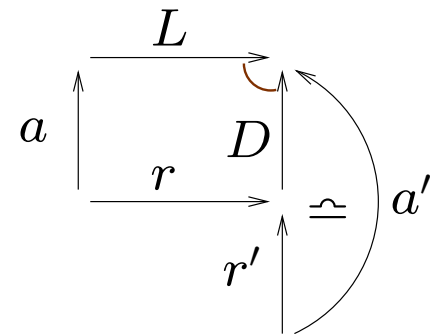
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The **full** transition system has *all* possible transitions. *But this is too many!*  
So consider only the transitions for which the pair  $(L, D)$  is minimal:

## STANDARD TRANSITIONS AND BISIMILARITY

**Standard contextual LTS**  $ST = (Ag_{ST}, Trans_{ST})$ :

- $Ag_{ST}$  has all ground arrows
- $Trans_{ST}$  has all **minimal** transitions.



**Standard bisimilarity:** the largest symmetric relation  $\sim$  such that

if  $a \sim b$  and  $b \xrightarrow{L} b'$ , with  $La$  defined, then  $a \xrightarrow{L} a'$  for some  $a' \sim b'$ .

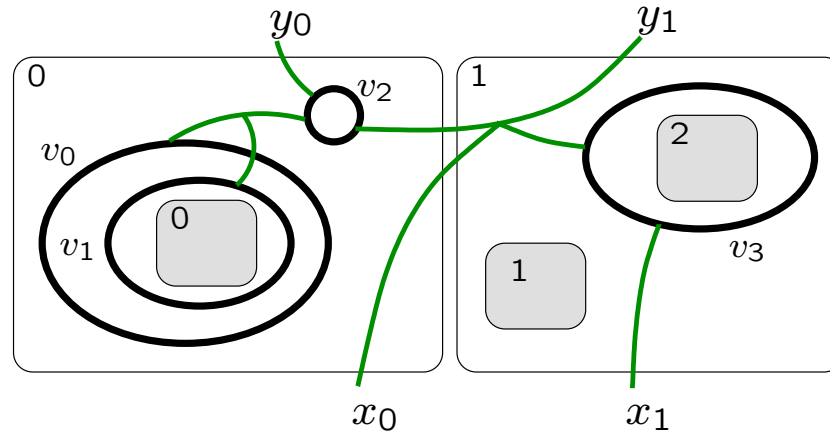
**THEOREM:** In any reactive system with RPOs, standard bisimilarity  $\sim$  is a congruence. (Leifer, Milner)



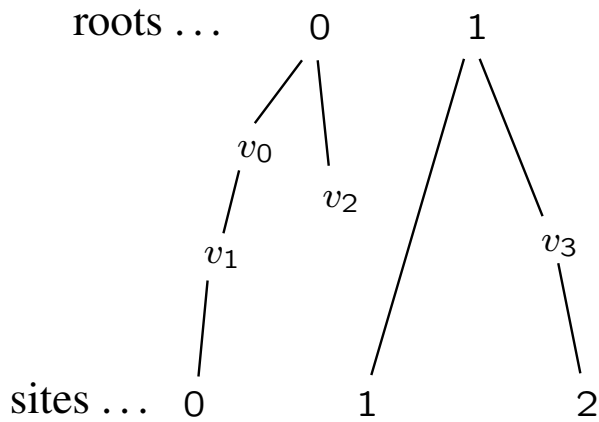
**Part III      LINK GRAPHS, SORTING and PETRI NETS**

# RESOLVING A BIGRAPH INTO PARTS

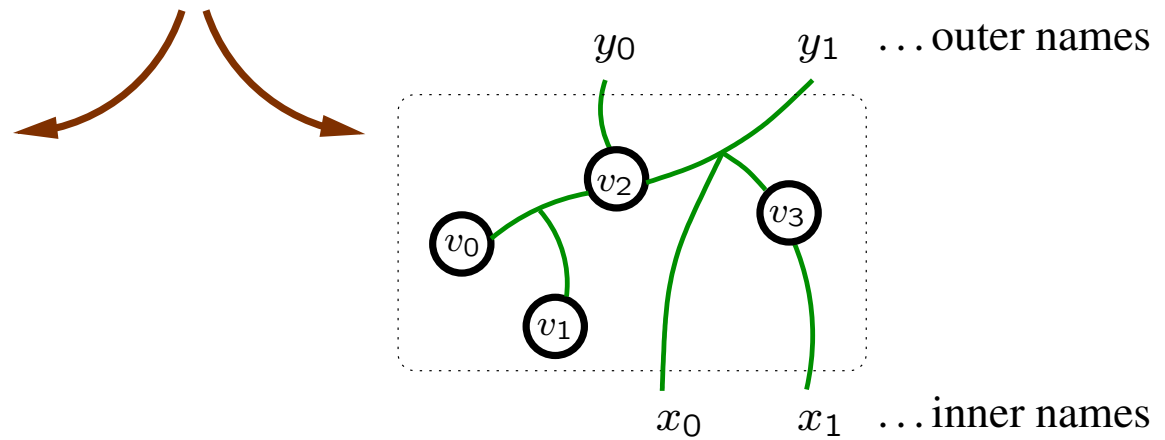
**bigraph**  
 $G: \langle m, X \rangle \rightarrow \langle n, Y \rangle$



**place graph**  
 $G^P: m \rightarrow n$



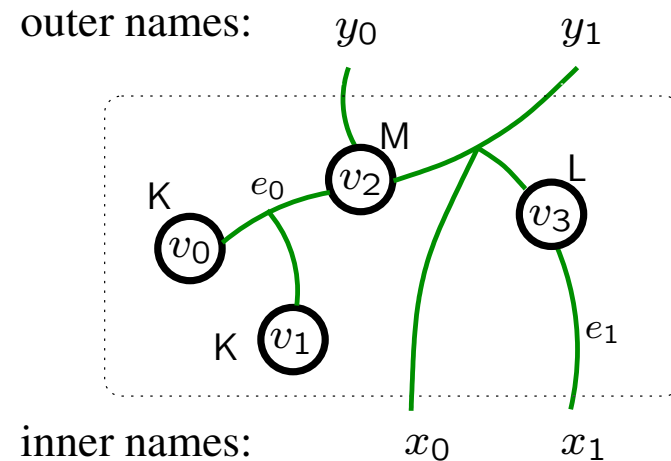
**link graph**  
 $G^L: X \rightarrow Y$



## A LINK GRAPH over signature $\mathcal{K} = \{K, L, M\}$

$$G = (V, E, ctrl, link) : X \rightarrow Y$$

- nodes  $V$  edges  $E$
- control map  $ctrl : V \rightarrow \mathcal{K}$
- ports  $P \stackrel{\text{def}}{=} \sum_{v \in V} ar(ctrl(v))$
- link map  $link : X \uplus P \rightarrow E \uplus Y$

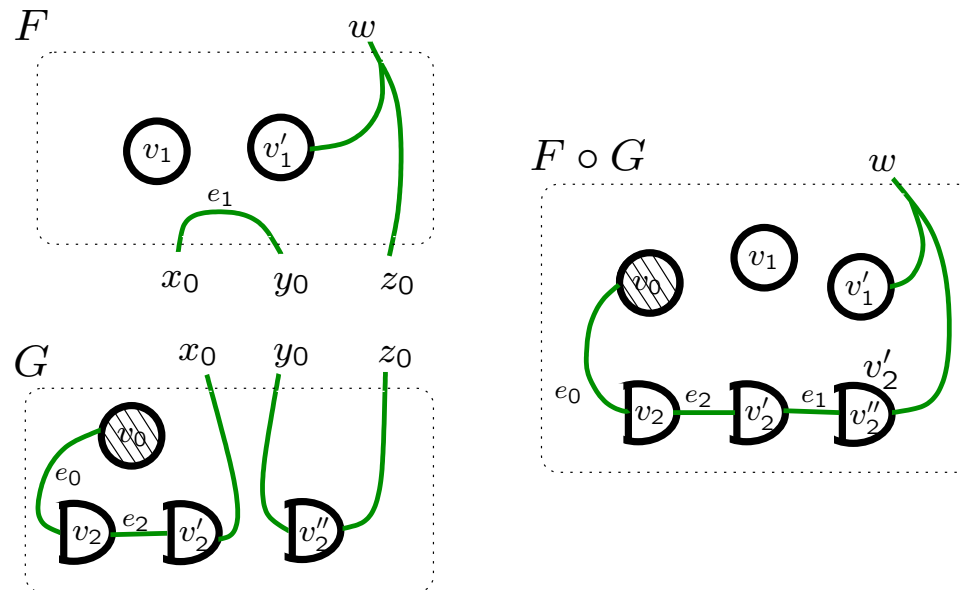


Inner names and ports are **points**.

Outer names are **open links**; edges are **closed links**.

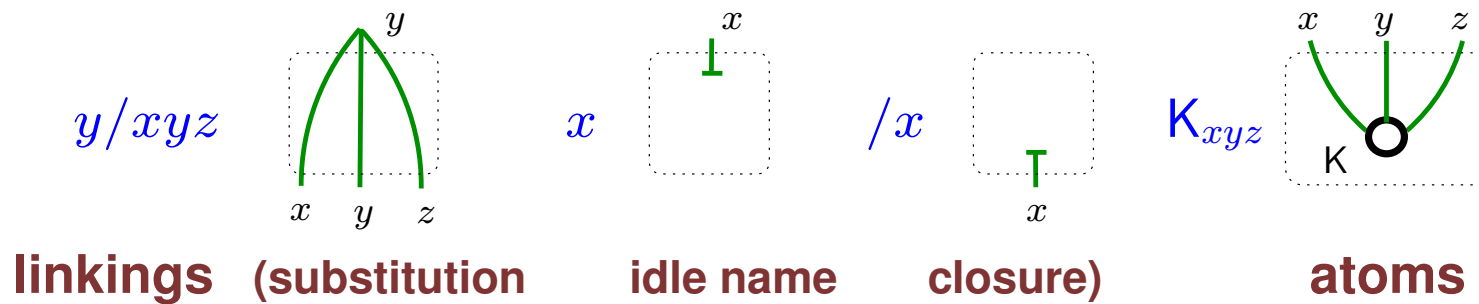
# THE S-CATEGORY OF LINK GRAPHS

**composition:** link the names



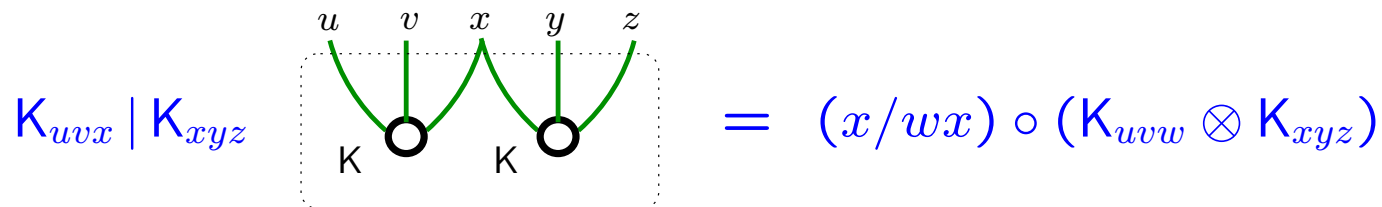
# ELEMENTARY LINK GRAPHS AND PARALLEL PRODUCT

All link graphs can be built by  $\circ$  and  $\otimes$  from **elements**:

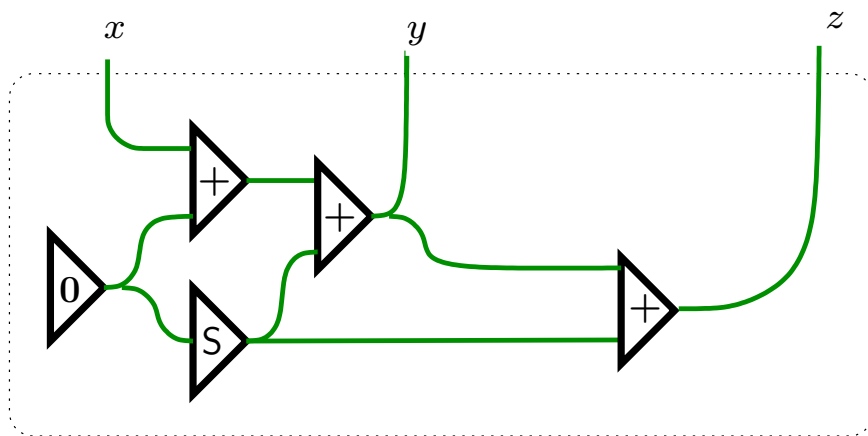


**Parallel product**  $F | G$  : derivable from tensor product and substitution.

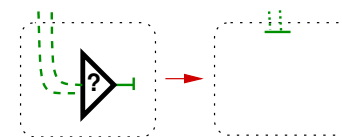
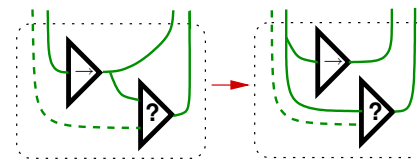
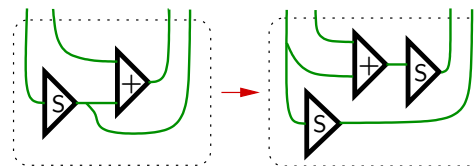
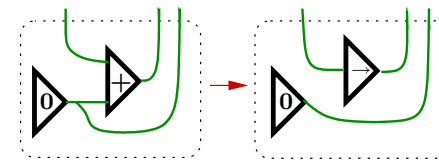
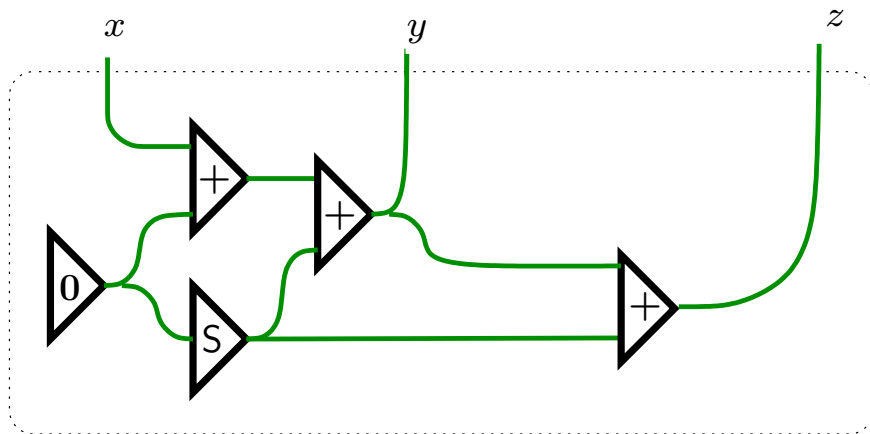
Like  $F \otimes G$ , but outer names may be shared. **For example:**



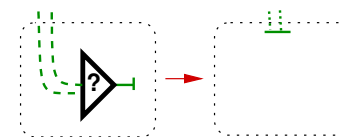
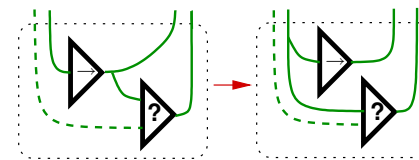
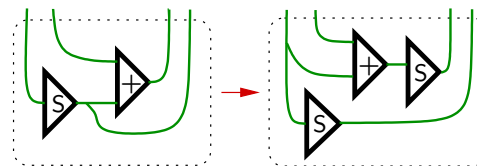
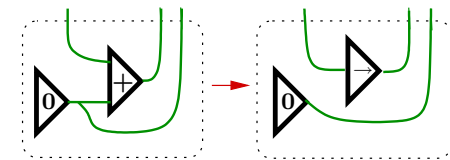
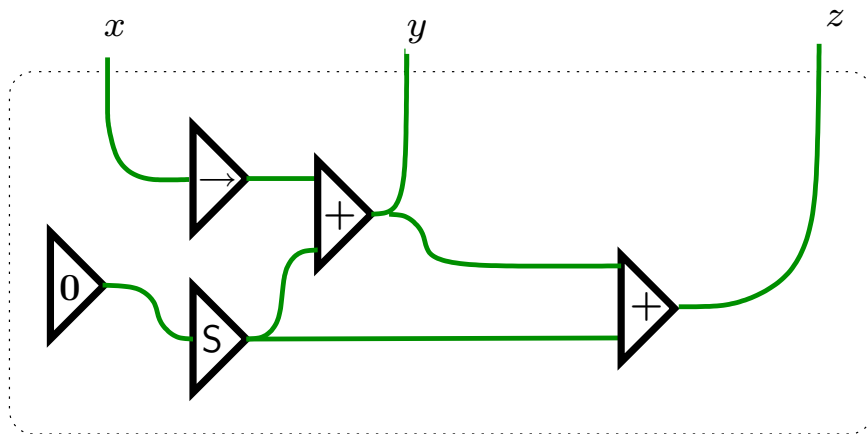
## AN ARITHMETIC NET



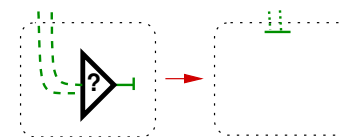
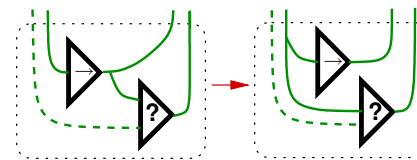
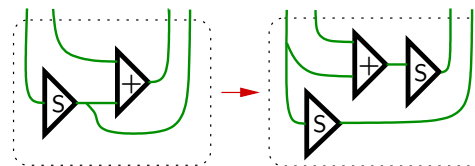
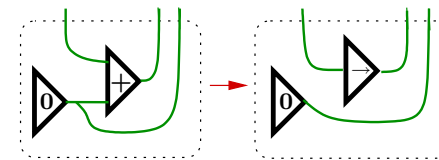
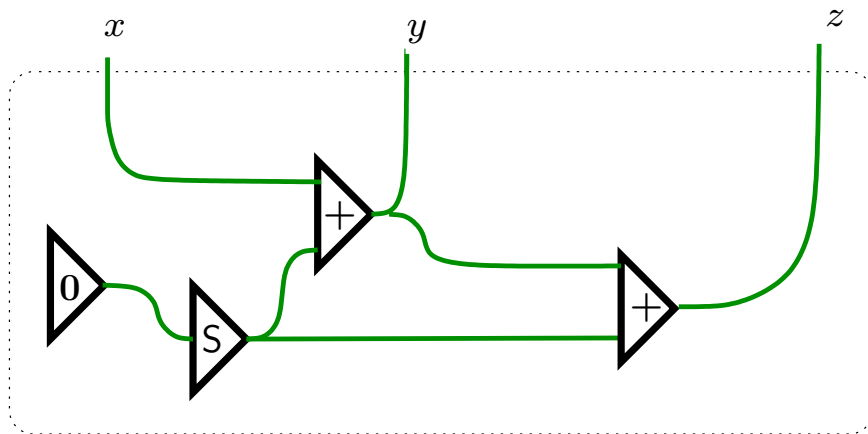
# AN ARITHMETIC NET and EVALUATION RULES



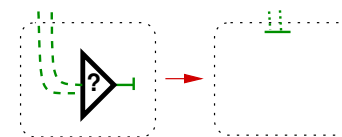
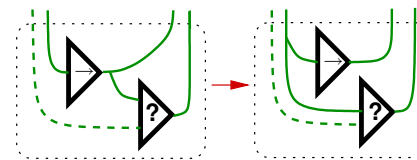
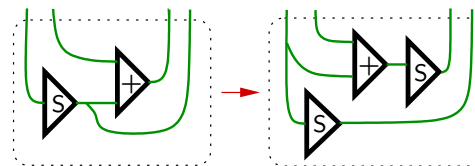
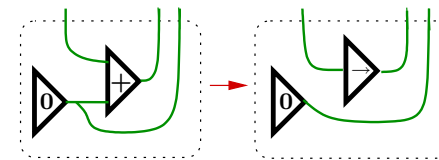
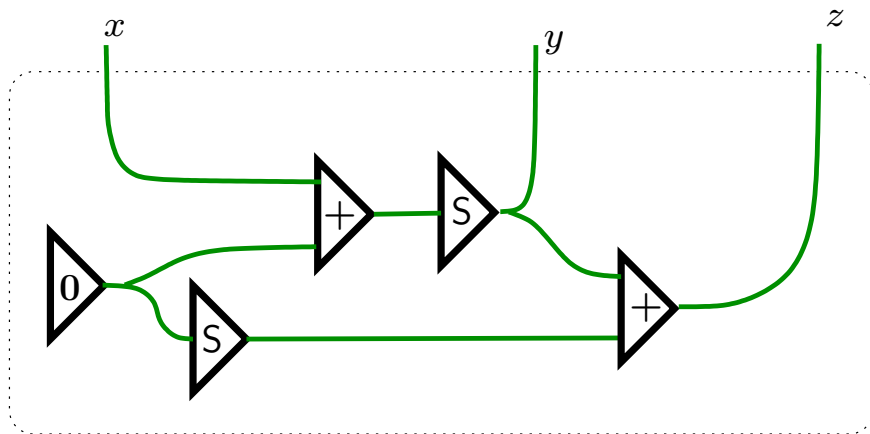
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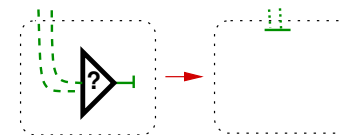
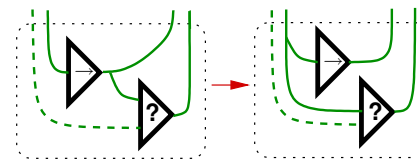
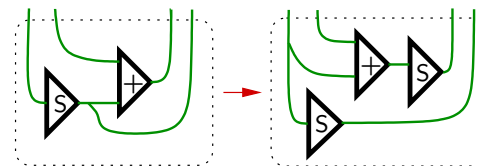
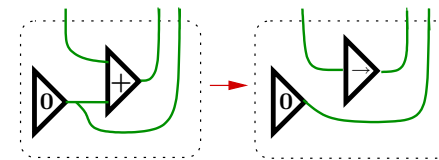
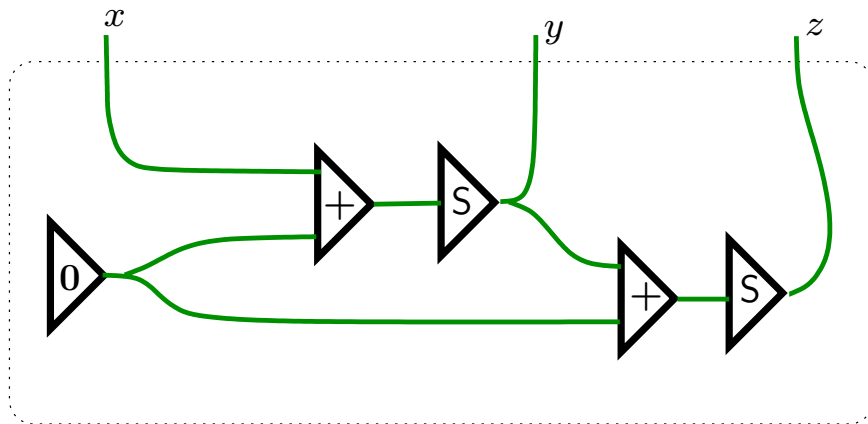
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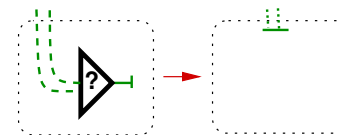
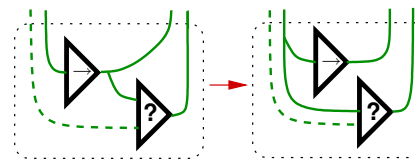
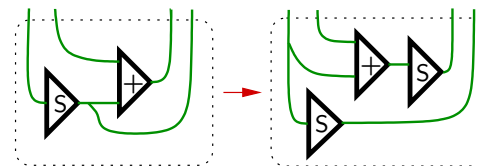
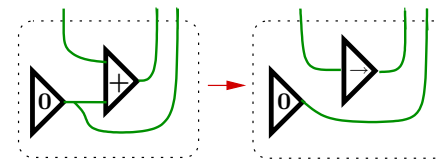
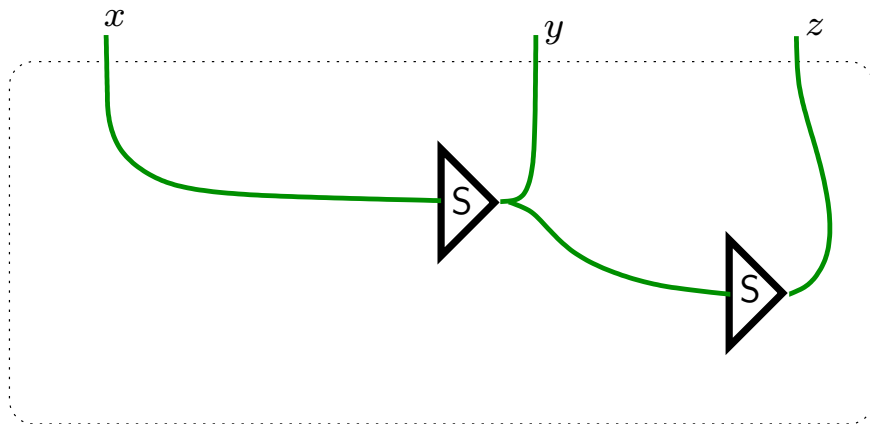
# AN ARITHMETIC NET and EVALUATION RULES



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# AN ARITHMETIC NET and EVALUATION RULES



## MANY-ONE SORTING

To distinguish well-formed link graphs, we often impose a **sorting** discipline on points and links. The signature  $\mathcal{K}$  determines the sort of every port.

In **many-one sorting** there are two sorts  $\{s, t\}$ ; source and target. A link may contain many number of **t**-points, and:

- a *closed* link has exactly one **s**-point;
- an *open s*-link has exactly one **s**-point;
- an *open t*-link has no **s**-points.

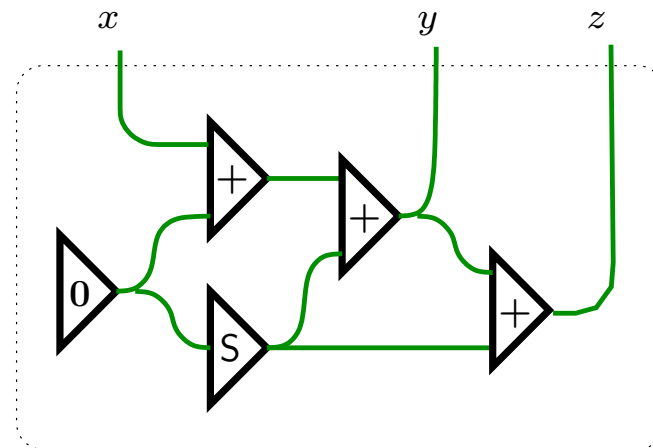
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Apply this to **arithmetic nets**:



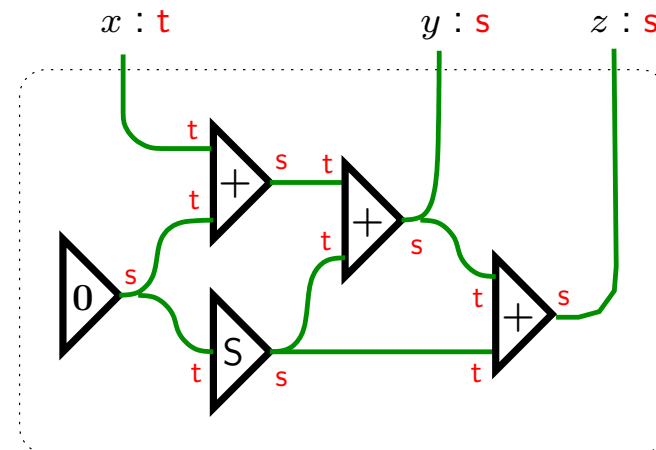
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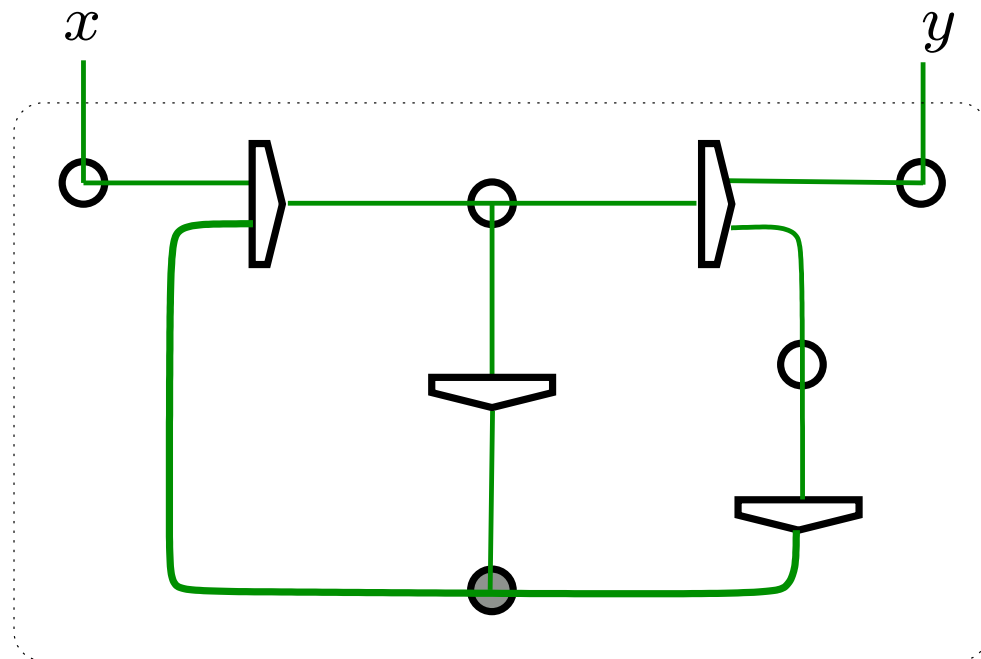
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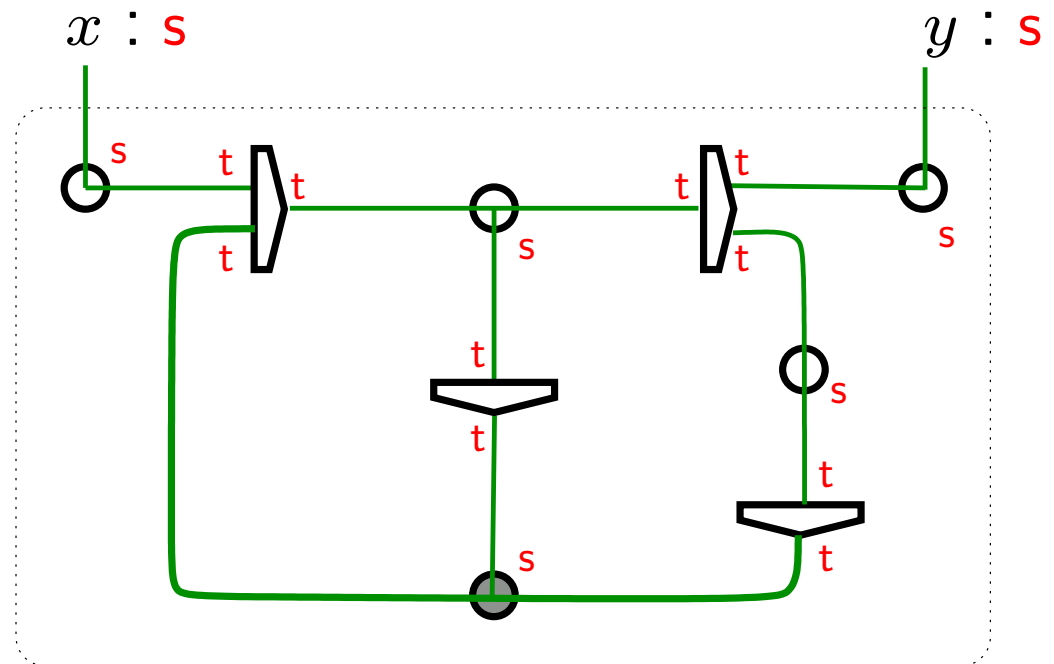
## MANY-ONE SORTING for PETRI NETS

To avoid having two conditions (or two places) contiguous, all **transition** ports have sort **t** and all **condition** ports have sort **s**.



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## 'RAW' LABELLED TRANSITION SYSTEM $\mathbf{A}(\mathcal{L})$

Equip  $\mathbf{A}$  with a **labelled transition system**  $\mathcal{L} = (\text{Ag}, \text{Lab}, \text{Trans})$  having **agents**  $\text{Ag}$  (ground arrows), **labels**  $\text{Lab}$ , and **transitions**  $\text{Trans}$  of form

$$a \xrightarrow{\ell} a'$$

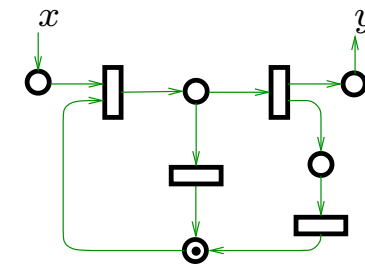
where  $a, a' \in \text{Ag}$  and  $\ell \in \text{Lab}$ .

In a transition, think of  $\ell$  as **observing** the agent  $a$ . This leads to equivalences such as **bisimilarity**, the largest symmetric relation  $\sim$  such that

$$\text{if } a \sim b \text{ and } b \xrightarrow{\ell} b', \text{ then } a \xrightarrow{\ell} a' \text{ for some } a' \sim b' .$$

## 'RAW' TRANSITIONS FOR CONDITION-EVENT NETS

As **agents**  $Ag$ , choose c-e nets with named conditions:



As **labels**  $Lab$ , choose

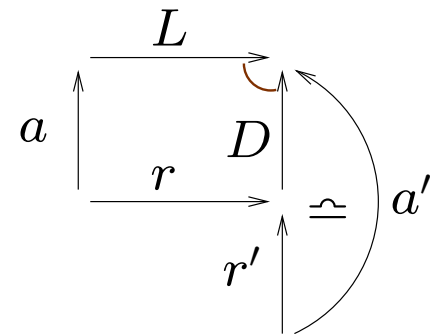
{	$+x$	mark the condition named $x$
	$-x$	unmark the condition named $x$
	$\tau$	perform a reaction internally .

This yields '**raw**' bisimilarity,  $\simeq$ . Is it a **congruence**? How does it compare with standard bisimilarity derived in the theory of link graphs?

## STANDARD TRANSITIONS AND BISIMILARITY

**Standard contextual LTS**  $ST = (Ag_{ST}, Trans_{ST})$ :

- $Ag_{ST}$  has all ground arrows
- $Trans_{ST}$  has all **minimal** transitions.



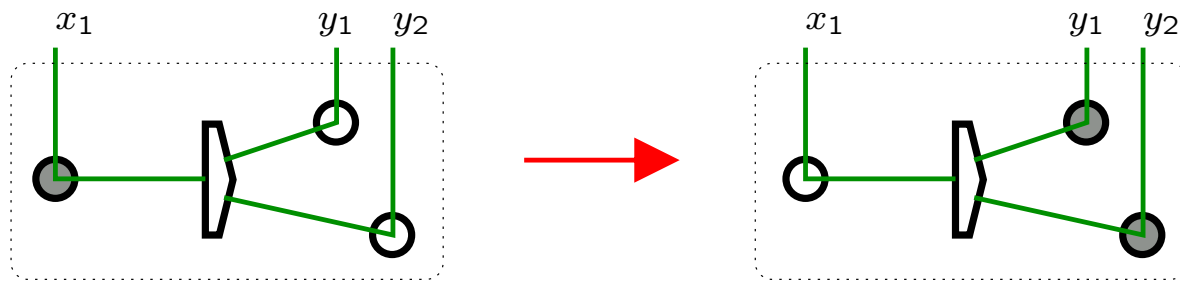
**Standard bisimilarity:** the largest symmetric relation  $\sim$  such that

if  $a \sim b$  and  $b \xrightarrow{L} b'$ , with  $La$  defined, then  $a \xrightarrow{L} a'$  for some  $a' \sim b'$ .

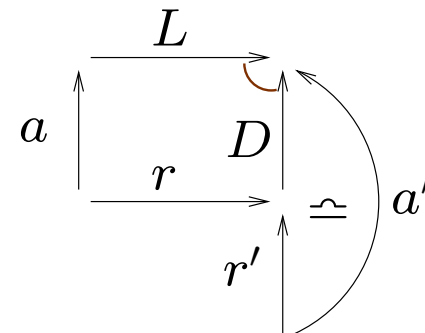
**THEOREM:** In any reactive system with RPOs, standard bisimilarity  $\sim$  is a congruence. (Leifer, Milner)

## DERIVED TRANSITIONS for PETRI NETS

A typical reaction rule is **simple**:



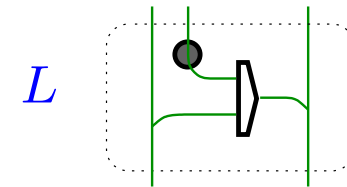
**THEOREM:** In a **simple** reactive system, for bisimilarity one need check only the **engaged** transitions: those where  $a$  and  $r$  share at least a node.



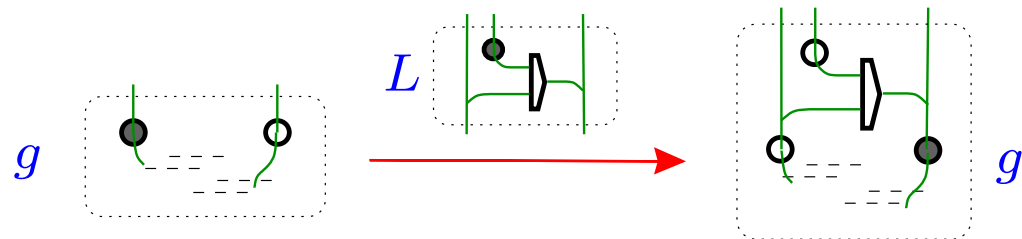
What do these transitions look like?

## ENGAGED TRANSITIONS and BISIMILARITY for PETRI NETS

A typical **label**  $L$  has a single event node; it expects the agent  $g$  to supply some pre- and post-conditions:



A typical  $L$ -transition:



These **derived** labels differ from the **raw** labels  $+x, -x, \tau$ . But:

**THEOREM :** **standard bisimilarity**  $\sim$  **and raw bisimilarity**  $\simeq$  **coincide.**

— and it follows that both are **congruences**.

## CONCLUSION

### Bigraphs explore:

- the modelling power of two or more **orthogonal structures**
- how to model **virtual space just like physical space**
- how to coordinate **different reactive systems**

For more detail, find links at

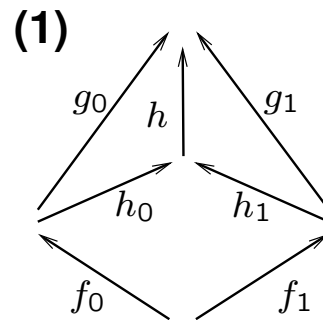
<http://www.cl.cam.ac.uk/users/rm135/> .



## RPOS AND IPOS IN AN S-CATEGORY

Write  $\vec{f}$  for  $f_0, f_1$ .

Call  $\vec{g}$  a **bound** for  $\vec{f}$  if  $g_0 f_0 = g_1 f_1$ .

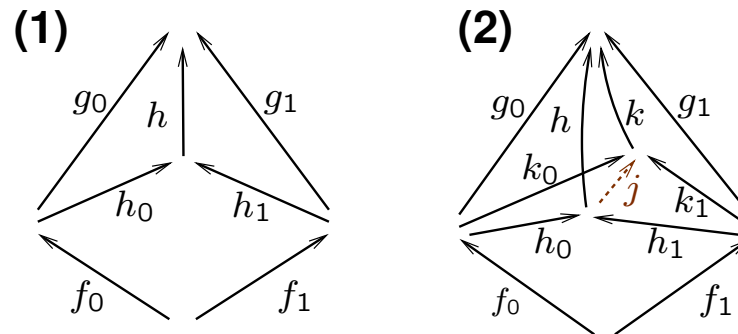


(1) A relative bound  $(\vec{h}, h)$  for  $\vec{f}$  to  $\vec{g}$ .

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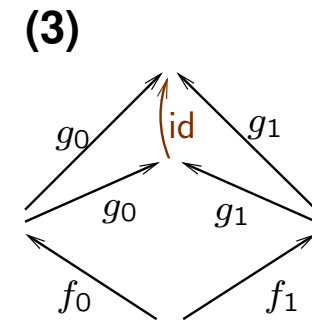
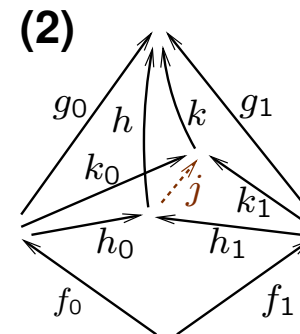
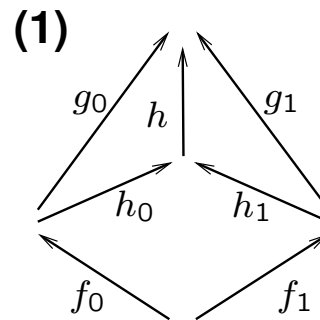
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(2) A **relative pushout (RPO)**  $(\vec{h}, h)$  for  $\vec{f}$  to  $\vec{g}$  : For any other relative bound  $(\vec{k}, k)$ , there is a unique mediator  $j$ .

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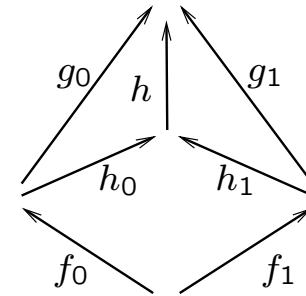
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(3) An **idem pushout (IPO)**  $\vec{g}$  for  $\vec{f}$ :  $(\vec{g}, \text{id})$  is an RPO for  $\vec{f}$  to  $\vec{g}$ .

## PROPERTIES OF RPOS and IPOS

- (1) An RPO for  $\vec{f}$  to  $\vec{g}$  is unique up to isomorphism.
- (2) If  $(\vec{h}, h)$  is an RPO for  $\vec{f}$ , then  $\vec{h}$  is an IPO for  $\vec{f}$ .
- (3) If  $\vec{h}$  is an IPO for  $\vec{f}$ , and an RPO exists for  $\vec{f}$  to  $\vec{g}$ , then  $(\vec{h}, h)$  is such an RPO.



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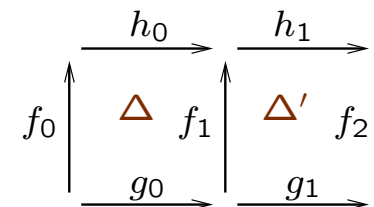
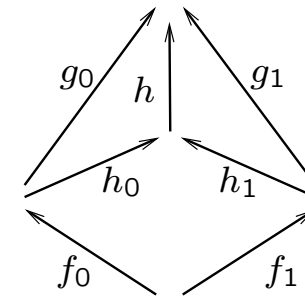
(1) An RPO for  $\vec{f}$  to  $\vec{g}$  is unique up to isomorphism.

(2) If  $(\vec{h}, h)$  is an RPO for  $\vec{f}$ , then  $\vec{h}$  is an IPO for  $\vec{f}$ .

(3) If  $\vec{h}$  is an IPO for  $\vec{f}$ , and an RPO exists for  $\vec{f}$  to  $\vec{g}$ , then  $(\vec{h}, h)$  is such an RPO.

- (4) • If  $\Delta$  and  $\Delta'$  are IPOs, so is the whole.  
 • If the whole and  $\Delta$  are IPOs, so is  $\Delta'$ .

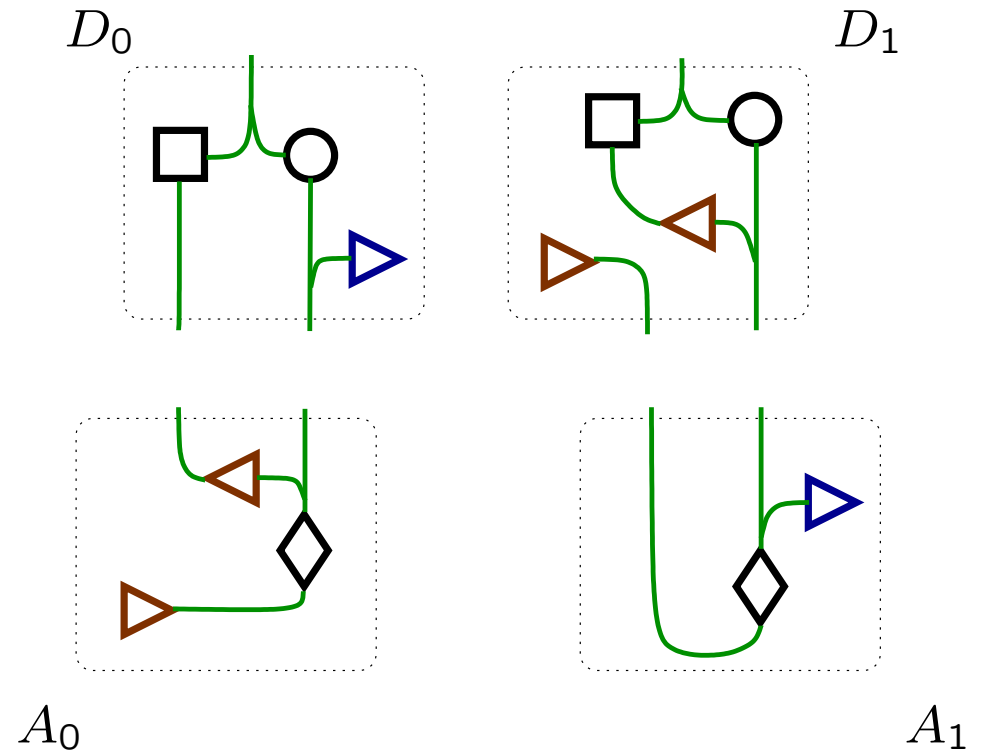
(when enough RPOs exist)



(5) In an s-category, if  $(\vec{f}, \vec{h}) \simeq (\vec{f}', \vec{h}')$  and one is an IPO, so is the other.

## AN RPO FOR LINK GRAPHS (1)

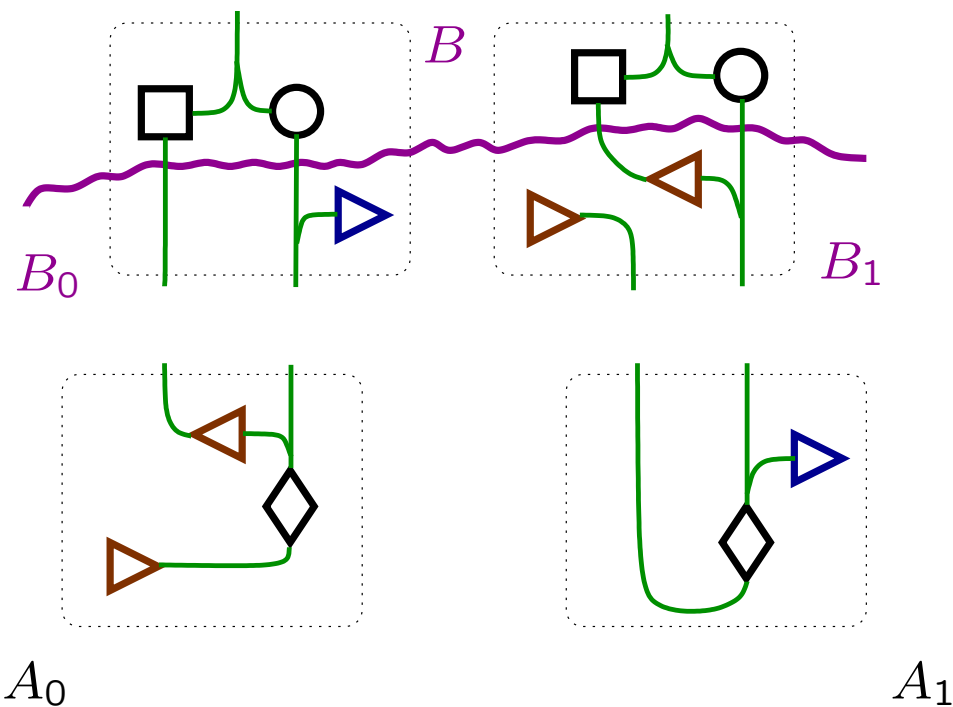
$(D_0, D_1)$  is a bound for  $(A_0, A_1)$



## AN RPO FOR LINK GRAPHS (2)

$(D_0, D_1)$  is a bound for  $(A_0, A_1)$

Truncate  $D_0, D_1$  to make  
an RPO  $(B_0, B_1, B)$



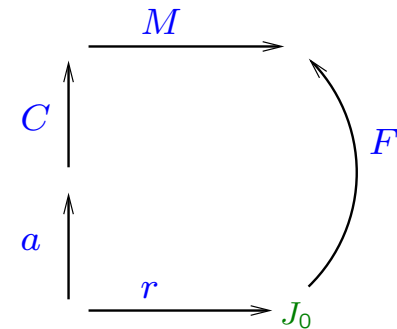
**THEOREM:** Link graphs have RPOs

## PROVING CONGRUENCE of BISIMILARITY

To prove that  $a \sim b \Rightarrow Ca \sim Cb$ , it is sufficient to show that  $\mathcal{S} = \{(Ca, Cb) \mid a \sim b\}$  is a bisimulation up to  $\simeq$ .

(1) Let  $a \sim b$ , and suppose  $\boxed{Ca \xrightarrow{M} a''}$   
with ground rule  $(r, r')$ . So  $a'' \simeq Fr'$ .

*We must find  $b''$  for which  $Cb \xrightarrow{M} b''$  and  $a'' \mathcal{S} b''$ .*

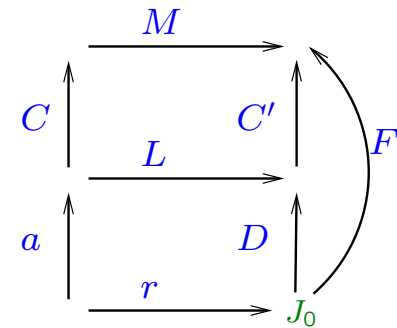


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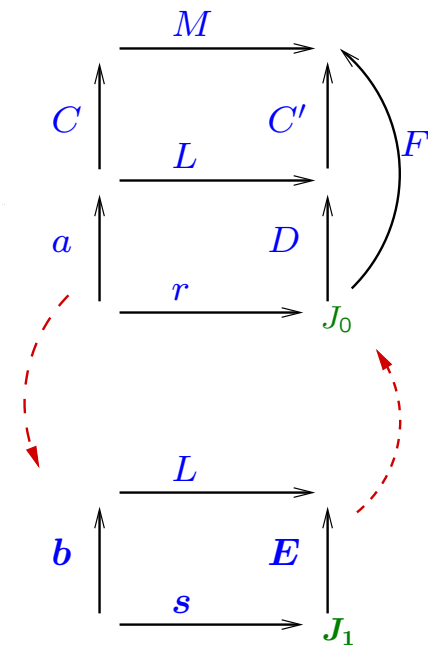
(2) Take the RPO of  $(a, r)$ .  
Then  $a \xrightarrow{L} a' \stackrel{\text{def}}{=} Dr'$ . So  $\boxed{a'' \simeq C'a'}$ .



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- (2) Take the RPO of  $(a, r)$ .  
Then  $a \xrightarrow{L} a' \simeq Dr'$ . So  $\boxed{a'' \simeq C'a'}$ .
- (3) From  $a \sim b$  deduce  $b \xrightarrow{L} b'$  with  $\boxed{a' \sim b'}$ .



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- (2) Take the RPO of  $(a, r)$ .  
Then  $a \xrightarrow{L} a' \simeq Dr'$ . So  $\boxed{a'' \simeq C'a'}$ .
- (3) From  $a \sim b$  deduce  $b \xrightarrow{L} b'$  with  $\boxed{a' \sim b'}$ .
- (4) Swap lower IPOs; deduce  $\boxed{Cb \xrightarrow{M} b'' \stackrel{\text{def}}{=} C'b'}$ . ■

