Research Note on Tarski's Elementary Geometry

Brandon Bennett
School of Computer Studies
University of Leeds, Leeds LS2 9JT, UK
brandon@scs.leeds.ac.uk

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Tarski (1959) has given the following axiomatisation of (2D) *elementary geometry* in terms of the two primitives, *betweenness* ($\mathsf{B}(x,y,z)$ (y is between x and z — including the cases where y is equal to x or z) and (quaternary) *equidistance* ($xy = zw$) (x is as distant from y as z is from w). (In fact, as we shall see in the next section, betweenness can itself be defined in terms of equidistance; but replacing B by such a definition would make the axioms much more cumbersome.)

B1 [Identity Axiom for Betweenness]
$\forall xy[\mathsf{B}(x,y,x) \rightarrow (x = y)$

B2 [Transitivity Axiom for Betweenness]
$\forall xyzu[(\mathsf{B}(x,y,u) \wedge \mathsf{B}(y,z,u)) \rightarrow \mathsf{B}(x,y,z)]$

B3 [Connectivity Axiom for Betweenness]
$\forall xyzu[(\mathsf{B}(x,y,z) \wedge \mathsf{B}(x,y,u) \wedge (x \neq y)) \rightarrow (\mathsf{B}(x,z,u) \vee \mathsf{B}(x,u,z))]$

B4 [Reflexivity Axiom for Equidistance]
$\forall xy[(xy = yx)]$

B5 [Identity Axiom for Equidistance]
$\forall xyz[(xy = zz) \rightarrow (x = y)$

B6 [Transitivity Axiom for Equidistance]
$\forall xyzuvw[((xy = zu) \wedge (xy = vw)) \rightarrow (zu = vw)]$

B7 [Pasch's Axiom]
$\forall txyzu\exists v[(\mathsf{B}(x,t,u) \wedge \mathsf{B}(y,u,z)) \rightarrow (\mathsf{B}(x,v,y) \wedge \mathsf{B}(z,t,v))]$

B8 [Euclid's Axiom]
$\forall txyzu\exists vw[(\mathsf{B}(x,u,,t) \wedge \mathsf{B}(y,u,z) \wedge (x \neq y)) \rightarrow (\mathsf{B}(x,z,v) \wedge \mathsf{B}(x,y,w) \wedge \mathsf{B}(v,t,w))]$

B9 [Five-Segment Axiom]
$\forall xx'yy'zz'uu'[((xy = x'y') \wedge (yz = y'z') \wedge (xu = x'u') \wedge (yu = y'u') \wedge \mathsf{B}(x,y,z) \wedge \mathsf{B}(x',y',z') \wedge (x \neq y)) \rightarrow (zu = z'u')]$

B10 [Axiom of Segment Construction (2D)]
$\forall xyuv\exists z[\mathsf{B}(x,y,z) \wedge (yz = uv)]$

B11 [Lower Dimension Axiom (2D)]
$\exists xyz[\neg\mathsf{B}(x,y,z) \wedge \neg\mathsf{B}(y,z,x) \wedge \neg\mathsf{B}(z,x,y)]$

B12 [Upper Dimension Axiom]
$\forall xyzuv[((xy = xv) \wedge (yu = yv) \wedge (zu = zv) \wedge (u \neq v)) \rightarrow (\mathsf{B}(x,y,z) \vee \mathsf{B}(y,z,x) \vee \mathsf{B}(z,x,y))]$

B13 [Elementary Continuity Axioms]
All sentences of the form:

$$\forall vw\ldots[\exists z\forall xy[\phi \wedge \psi \rightarrow \mathsf{B}(z,x,y)] \rightarrow \exists u\forall xy[\phi \wedge \psi \rightarrow \mathsf{B}(x,u,y)]]$$

where ϕ stands for any formula in which the variables x, y, w, $\ldots$, but neither y nor z nor u, occur free, and similarly for ψ, with x and y interchanged.

B13' [Weak Continuity Axiom]

$\forall xyzx'z'u\exists y'[((ux = ux') \wedge (uz = uz') \wedge \mathsf{B}(u,x,z) \wedge \mathsf{B}(x,y,z)) \rightarrow$
$((uy = uy') \wedge \mathsf{B}(x',y',z'))]$

This axiom set fixes the dimension of the space to two. This is achieved by means of axioms **B**11 and **B**12. **B**11 is true in dimension 2 or greater, whilst **B**12 is true in dimension 2 or lower. All the other axioms are true in any dimension. In fact, the theory can be amended to determine Euclidean space of any given dimension. This can be achieved by means of a 1st-order formula of the following form:

$$\exists\, x_0 \ldots x_n \left[\bigwedge_{0 \le i \ne j \ne k \le n} (x_i \ne x_j \wedge (x_i x_j = x_j x_k)) \right]$$
$$\wedge\ \neg\exists\, x_0 \ldots x_{n+1} \left[\bigwedge_{0 \le i \ne j \ne k \le n+1} (x_i \ne x_j \wedge (x_i x_j = x_j x_k)) \right]$$

where $d(x,y)$ is the distance between x and y.

Definitions from the Ternary Equidistance Primitive

The sequence of definitions given below show how starting from the fundamental ternary relation $xy = yz$, which is true when two points, x and z, are equidistant from a third point, y, many other simple geometrical relations can be introduced. In these definitions, the juxtaposition xy of two variables x and y is intended to refer to the distance between these two points. Thus $xy \le yz$ is a predicate which holds in case y is closer to x than to z. The other relations are: $\mathsf{B}(x,y,z)$ — y is between x and z (including the case where y is identical with either x or z); $\mathsf{L}(x,y,z)$ — x, y and z are *collinear*; and $\mathsf{M}(x,y,z)$ — y is the mid-point between x and z.

The relation $xy = yz$ is fundamental because it relates the centre point of a sphere to any pair of surface points. For a 2-dimensional figure, the truth of this relation for any three points can be determined by means of a compass. The relations $\mathsf{B}(x,y,z)$ and $xy = zw$ are taken as primitives in Tarski's elementary geometry. A proof that the quaternary relation $xy = zw$ is definable in terms of the ternary $xy = yz$ is originally due to Pieri (1899). The following definitions showing how this can be done (together with further discussion of primitive notions in geometry) can be found in (Royden 1959).

$$\begin{aligned}
xy \le yz &\equiv_{def} \forall w[yw = wz \rightarrow \exists u[xu = uy \wedge uy = yw]] \\
\mathsf{B}(x,y,z) &\equiv_{def} \forall w[(wx \le xy \wedge wz \le zy) \rightarrow w = y] \\
\mathsf{L}(x,y,z) &\equiv_{def} \mathsf{B}(x,y,z) \vee \mathsf{B}(y,x,z) \vee \mathsf{B}(x,z,y) \\
\mathsf{M}(x,y,z) &\equiv_{def} \forall w[(\mathsf{L}(w,x,y) \wedge xy = yw) \leftrightarrow (w = x \vee w = z)] \\
(wx = yz) &\equiv_{def} \exists u \exists v[\mathsf{M}(w,u,y) \wedge \mathsf{M}(x,u,v) \wedge vy = yz]
\end{aligned}$$

References

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