A proximity approach to some region-based theories of space

by

Dimiter Vakarelov (Sofia, Bulgaria)
Georgi Dimov (Sofia, Bulgaria)
Ivo Düntsch (Ontario, Canada) and
Brandon Bennett (Leeds, England)

Abstract

This paper is a continuation of [41]. The notion of *local connection algebra*, based on the primitive notions of *connection* and *boundedness*, is introduced. It is slightly different but equivalent to Roeper's notion of *region-based topology* ([31]). The similarity between the *local proximity spaces* of Leader ([25]) and local connection algebras is emphasized. Machinery, analogous to that introduced by Efremovič ([17],[18]), Smirnov ([33]) and Leader ([25]) for proximity and local proximity spaces, is developed. This permits us to give new proximity-type models of local connection algebras, to obtain a representation theorem for such algebras and to give a new shorter proof of the main theorem of Roeper's paper [31]. Finally, the notion of *MVD-algebra* is introduced. It is similar to Mormann's notion of *enriched Boolean algebra* ([28]), based on a single mereological relation of *interior parthood.* It is shown that MVD-algebras are equivalent to local connection algebras. This means that the connection relation and boundedness can be incorporated into one, mereological in nature relation. In this way a formalization of the Whiteheadian theory of space based on a single mereological relation is obtained.

2000 *Mathematics Subject Classification:* Primary 06E15, 54E05; Secondary 54D45, 06E99.

Key words and phrases. Pointless geometry, connection relation, connection algebra, local connection algebra, MVD-algebra, proximity and local proximity spaces.

---

[1] Co-operation for this paper was supported by EU COST Action 274 "Theory and Applications

1 Introduction

The roots of *region-based* theories of space, known also as *pointless geometries*, go back to De Laguna [10] and to Whitehead [43]. The distinctive feature of the pointless approach is that it considers the classical notion of *point* (and the related notions of *line* and *surface*) as too abstract to be taken as basic primitive notions in the theory of space. Instead of points, these theories take as primitives the more intuitive notion of a *spatial region* and some binary relations between regions, for instance the so called *connection relation*. Then points (lines and surfaces) are (second-order) definable as certain sets of regions and in this way they become some of the complex notions of the theory. Besides the early work of Tarski [38] and Leonard and Goodman [26], other papers related to Whitehedian region-based approach to the theory of space include Grzegorczyk [20], Clarke [7, 8], Biacino and Gerla [3, 4], Roeper [31], Mormann [28], Bennett et al. [2] and Bennett [1]. We refer to the paper of Gerla [19] for a good survey on pointless geometry.

The fact that mereological relations such as "part-of", "overlap", "non-tangential inclusion" and others can be defined in terms of the connection relation relates some pointless geometries to the field of mereology [27] and to its fusion with topology — mereotopology [42]. The later is closely related to *naive* or *qualitative physics* introduced by Hayes [21] and in particular to its subfield *Qualitative Spatial Reasoning* (QSR) [9]. A very popular system in this field is the *Region Connection Calculus* (RCC) introduced in [30]. It has been realized that searching for models of mereological and RCC-like systems, methods of lattice theory and the theory of relation algebras can be successfully employed, see e.g. [12, 14, 34, 35, 15, 13, 28]. Mormann [28] has shown that the theory of continuous lattices also has applications in the region-based theory of space.

It is remarkable, however, that in the fast growing field of mereology, mereotopology and pointless theory of space, little attention has been paid to the theory of proximity spaces, the only exception being the early paper by Šwarc [36], in which the notion of *proximity distributive lattice* was introduced, and a topological representation theorem was proved. It was mentioned there, that the paper can be considered as an attempt to build a pointless analogue of the notion of proximity space. The paper had been written under the direction of the Russian Professor V. A. Efremovič, the founder of the theory of proximity spaces [18]. This makes it clear that the possibility to build a pointless analogue to proximity theory, and in general, to topology, had been known to its originators.

---

of Relational Structures as Knowledge Instruments " (TARSKI), www.tarski.org

D. Vakarelov and I. Düntsch have been partially sponsored by the NATO Collaborative Linkage Grant PST.CLG. 977641.

B. Bennett's work was partially supported by the EPSRC under grant GR/M56807.

Roughly speaking, a proximity space is a non-empty set X with a binary relation δ between subsets, called a *proximity relation*, with the intuitive meaning that $A\delta B$ holds, when "A is near B" in some sense. For instance, in the proximity spaces arising from metric spaces, $A\delta B$ means that the distance between the subsets A and B is zero. The proximity relation satisfies axioms that are identical with some of the typical axioms of the connection relation, which makes the theory directly applicable to systems based on the connection relation. Each proximity space determines a natural topology with nice properties, and the theory possesses deep results, rich machinery and tools. The main reference on proximity spaces is the book by S.A. Naimpally and B.D. Warrak [29].

With this paper we hope to demonstrate the fruitfulness of the theory of proximity spaces to certain formalizations of region-based theory of space. The paper is a continuation and a revised version of [41], where the notion of *connection algebra* has been introduced and a representation theorem of connection algebras in proximity spaces has been proved. Connection algebras are Boolean algebras whose elements are meant to formalize spatial regions with an additional relation C of "connection" between regions. Standard examples of connection algebras are the algebras of closed or open regions of certain topological spaces. The standard meaning of the connection relation between closed regions commonly accepted in the literature is the following:

$$xCy \text{ iff the closed regions } x \text{ and } y \text{ share a common point,}$$

and the standard meaning of the connection relation between open regions is:

$$xCy \text{ iff } cl(x) \cap cl(y) \neq \emptyset,$$

where cl is the operation of topological closure.

Proximity spaces offer another natural modelling of the connection relation:

$$xCy \text{ iff } x\delta y$$

where x and y are closed (or open) regions in a proximity space and δ is the proximity relation on the space. The representation theorem in [41] is just of this proximity kind and this is one of the main novelties of [41]. In the present paper we prove a stronger version of the representation theorem, proving directly that each connection algebra can be isomorphically embedded into the connection algebra of a proximity space, corresponding to a compact Hausdorff space. Note that such spaces admit a unique proximity relation

$$A\delta B \text{ iff } cl(A) \cap cl(B) \neq \emptyset,$$

having the standard topological meaning of the connection relation between open regions. We prove also that there exists a bijective correspondence between the class of all (up to isomorphism) complete connection algebras and the class of all (up to homeomorphism) compact Hausdorff spaces. The representation construction simulates the Leader's proof of the Smirnov Compactification Theorem (see [29]). Most of the proofs and constructions in this paper can be considered as lattice-theoretic versions of previously known proofs and constructions in the theory of proximity spaces.

The next thing we deal with is an application of the proximity approach to the system of *region-based topology* introduced by Roeper in [31]. Roeper's notion of region-based topology is a region-based theory of space that is closely related to our notion of connection algebra — it is a Boolean algebra with a connection relation and an additional one-place predicate of *limitedness.* Together with the connection relation, limitedness is another important property of spatial regions. For instance, in metric or Euclidean spaces the notion of limited region coincides with the notion of bounded region (region contained in some sphere). The main representation theorem is that there exists a bijective correspondence between the class of all (up to isomorphism) region-based topologies and the class of all (up to homeomorphism) locally compact Hausdorff spaces. Roeper proves this theorem by developing his own techniques without using much of the known tools and existing methods in topology and proximity theory.

The axioms of "region-based topology" almost coincide with the axioms of *local proximity spaces* introduced by Leader [25]. This similarity prompts us to introduce a new name for region-based topologies and to call them *local connection algebras.* The similarity between local proximity spaces and local connection algebras makes it possible to obtain the main representation theorem of Roeper by application of the methods of local proximity spaces. The main results for local proximity spaces are obtained by applying the corresponding results for (Efremovič) proximity spaces. With this in mind, we obtain a representation theorem for local connection algebras in local proximity spaces by an application of the representation theorem for connection algebras in proximity spaces. In this way we also obtain a shorter proof of the main theorem of Roeper.

Another work devoted to region-based theory of space is Mormann's paper [28]. His system, called *enriched Boolean algebra*, is similar to connection algebras, but instead of the connection relation, it contains another relation between (open) regions, called *interior parthood.* One of the main aims of Mormann's paper is to show that Whiteheadian theory of space can be built up on the base of the single relation of interior parthood, considered as a "purely mereological relation" ([28], p. 37). In a discussion with Roeper (p. 52) he claims that the relations of connection and limitedness are "non-mereological". We use here without discussion Mormann's terminology, although it seems that all such relations should be considered as "*mereotopological*". Note however that Mormann's notion of interior parthood is different from the corresponding notion in local connection algebras. Mormann's definition in the intended semantics — open regions in locally compact Hausdorff spaces, is the following:

$$x \ll y \text{ iff } cl(x) \subseteq y \text{ and } cl(x) \text{ is compact, where } x, y \text{ are open regions.}$$

The difference with the corresponding definition in Roeper's paper [31] is in the requirement of compactness of $cl(x)$ which introduces some asymmetry between x and y. If we define connection by the standard formula "xCy" iff "not $x \ll y^*$" then the above asymmetry implies that the connection relation is not a symmetric one as it should be. Despite this difference, the main representation theorem for Mormann's system takes the same form as Roeper's. Unfortunately, the representation theorem presented in Mormann's paper [28] is not true (see Example 6.2 here). However it becomes true if one adds an extra axiom to those which an enriched Boolean algebra has to satisfy. The obtained new notion, introduced by the first two authors of the present paper, is called an *MVD-algebra* (Mormann-Vakarelov-Dimov-algebra). We prove that MVD-algebras are equivalent to local connection algebras. Hence the representation theorem for local connection algebras is valid also for MVD-algebras. In this way we extend the proximity approach to MVD-algebras. The representation result for MVD-algebras implies also that one of the main aims of the Mormann's paper — to formalize the Whitehedian theory of space on the base of a single mereological relation, is realized. It is shown in this way that the modified Mormann's notion of interior parthood incorporates in itself both of connection and boundedness, which is a quite unexpected fact.

Applications of proximity spaces to similar problems can be found in [40] and [11]. In [11] proximity spaces are used to formalize the notions of local and global similarity relations between objects in certain information systems. A local similarity relation has a meaning analogous to the overlapping relation in mereology, and a global similarity relation is interpreted by the proximity relation. This shows another possible approach to the theory of spatial relations — to use the representation theory of similarity relations (or, more generally, informational relations) in information systems (see Vakarelov [39] for references).

The rest of the paper is organized as follows. In section 2 we introduce the notion of connection algebra and in section 3 we list some basic facts about proximity spaces. We prove here that the Boolean algebra of closed regions of each proximity space forms a connection algebra with respect to the proximity relation. Section 4 is devoted to the representation theorem for connection algebras. The proof is analogous to the proof of the Stone representation theorem for Boolean algebras and to Leader's proof of the Smirnov Compactification Theorem. In section 5 we give the definition and some facts about local proximity spaces and introduce the notion of local connection algebra, which, though slightly different in formulation, is equivalent to Roeper's notion of "region-based topology" [31]. The changes have been made in order to fit well with the definition of local proximity space. It is proved here that the closed regions of a local proximity space determine a local connection algebra. The main result in this section is the representation theorem for local connection algebras, which is deduced from the corresponding theorem for connection algebras. The main idea of the proof is a lattice-theoretic parallel with Leader's theorem for local proximity spaces [29, 25]. In this way we show how the theory of the more simple notion of connection algebra can be used to obtain the main representation theorem for local connection algebras. In section 6 we introduce the notion of an *MVD-algebra*, which is similar to the Mormann's notion of enriched Boolean algebra. The formal equivalence of the notions of MVD-algebras and local connection algebras is proved and a representation theorem for MVD-algebras is obtained. Section 7 is devoted to some concluding remarks.

If (X, τ) is a topological space, we denote by $cl(A)$ the topological closure in (X, τ) of the subset A of X. We don't assume that completely regular topological spaces and normal topological spaces are T_1-spaces. When they are supposed to be T_1-spaces, we use the terms, respectively, "Tychonoff spaces" and "T_4-spaces".

If X is a set then by $Exp(X)$ we denote the powerset of X.

For all undefined notions and notations in the paper see [32] or [24] for Boolean algebras, [29] for proximity spaces and [16] for topology.

2 Connection algebras

Definition 2.1 An algebraic system $A = (B, 0, 1, \vee, \wedge, ^*, C)$ is called a *connection algebra* (abbreviated as CA), if $(B, 0, 1, \vee, \wedge, ^*)$ is a Boolean algebra (where the operation "complement" is denoted by "*") and C is a binary relation on B, satisfying the following axioms (we will write "$-C$" for "not C"):

(C1) If $x \wedge y \neq 0$ then xCy;

(C2) If xCy then $x, y \neq 0$;

(C3) xCy implies yCx;

(C4) $xC(y \vee z)$ iff xCy or xCz;

(C5) If $x(-C)y$ then $x(-C)z$ and $y(-C)z^*$ for some $z \in B$;

(C6) If $x \neq 1$ then there exists $y \neq 0$ such that $y(-C)x$.

Usually, we shall just write (B, C) for a connection algebra. The elements of B will be called (spatial) *regions* and the relation C a *connection relation.*

We will say that two connection algebras (B_1, C_1) and (B_2, C_2) are *isomorphic* iff there exists a Boolean isomorphism $f : B_1 \longrightarrow B_2$ such that, for each $a, b \in B_1$,

aC_1b iff $f(a)C_2f(b)$.

Note that our definition of CA differs from that of "Boolean connection algebra" of [34].

Fact 2.2 *Let (B, C) be a connection algebra. Then the following is true:*
$(C6')$ If $x \not\leq y$ then zCx and $z(-C)y$ for some $z \in B$.

Proof. We have $x \not\leq y \leftrightarrow x \wedge y^* \neq 0 \leftrightarrow x^* \vee y \neq 1$. Now, by (C6), there exists an element z of $B \setminus \{0\}$ such that $z(-C)(x^* \vee y)$. Then (C4) implies that $z(-C)y$ and $z(-C)x^*$. Supposing that $z(-C)x$, we obtain, by (C4), that $z(-C)1$, which means, according to (C1) and (C2), that $z = 0$, which is a contradiction. Hence, zCx. □

Remark 2.3 Obviously, $(C6')$ implies (C6) (indeed, $x \neq 1$ means that $1 \not\leq x$; hence there exists an $y \in B$ such that $yC1$ and $y(-C)x$; by (C2), $yC1$ implies $y \neq 0$). So, in the definition of a connection algebra, one can substitute (C6) with $(C6')$, as it is done in [41]. We take the axiom (C6) from [34].

Remark 2.4 Axiom (C6) is an extensionality axiom, since it implies, in the presence of the other axioms, the statement $(C6')$ and one easily obtains from the later (see Lemma 2.5(3) bellow) that

$$x = y \Longleftrightarrow (\forall z \in B)[zCx \Longleftrightarrow zCy].$$

The following lemma collects some easy properties of the connection relation:

Lemma 2.5 *Suppose that (B, C) is a connection algebra. Then:*

1. *If xCy, $x \leq x'$ and $y \leq y'$ then $x'Cy'$,*
2. *xCx iff $x \neq 0$,*
3. *$x \leq y$ iff $(\forall z)(zCx$ implies $zCy)$.*

2.6 We define *non-tangential inclusion* $\ll$ by

$$x \ll y \Longleftrightarrow x(-C)y^*.$$

This relation has different names in the literature: "well-inside relation", "well below", "interior parthood", "non-tangential proper part" or "deep inclusion". The relations C and $\ll$ are inter-definable, and the axiomatization of C can be equivalently rewritten in terms of non-tangential inclusion (see [41]). It is easy to see that axiom (C5) can be expressed equivalently in the following form:

$(C5')$ If $x \ll z$ then $x \ll y \ll z$ for some $y \in B$.

The following is equivalent to (C6):

$(C6'')$ If $x \neq 0$ then there exists $y \neq 0$ such that $y \ll x$.

Our connection algebras are strongly related to the structures of [20] and [7]. All of our axioms, except (C5), are either axioms or theorems in the Grzegorczyk system. Grzegorczyk assumes B to be a complete Boolean algebra but we have decided to drop this assumption in order to obtain a first-order notion of connection algebra. Grzegorczyk also assumes two non-trivial axioms in his system, containing the definable notion of point. In pointless geometry, the definition of *point* is one of the central problems, and it is defined (in different ways) as a collection (or sequence) of regions. The inclusion of the notion of point in the set of axioms, as Grzegorczyk did, makes the system complicated and not first-order. Biacino and Gerla ([4]) have discussed the problem of equivalence of the notion of point in the systems of Grzegorczyk ([20]) and Whitehead ([43]). They proved that the two notions are equivalent if the Grzegorczyk system is enriched with the axiom (C5) above. They also noted that this axiom is satisfied in the models of the connection relation in "good" topological spaces — for instance in all normal spaces and, in particular, in n-dimensional Euclidean spaces. This is one of the motivations for us to include (C5) in our definition of CA. In this way we obtain a region-based theory of space for which a good topological representation theorem can be proved without assuming axioms containing the notion of point.

3 Proximity spaces

In this Section we recall the basic definitions and properties of proximity spaces. The proofs of the facts concerning proximities mentioned here can be found in [29]. It will turn out that connection algebras can be represented by the regular closed subsets of proximity spaces.

3.1 Let X be a non-empty set. A binary relation δ on the powerset $Exp(X)$ of X is called a *basic proximity* or simply *proximity*, if it satisfies (C1)–(C4) on the Boolean algebra $(Exp(X), \emptyset, X, \cup, \cap, \setminus)$; the pair (X, δ) is called a *basic proximity space* or simply *proximity space*. When x is a point of X, we write $x\delta A$ in place of $\{x\}\delta A$. A basic proximity is called *separated* if it satisfies

(SP) $x\delta y$ implies $x = y$.

In such a case the pair (X, δ) is called *separated basic proximity space*.

If M is a subset of X then the *restriction* δ_M of δ to M is defined as follows: for $A, B \subseteq M$, $A\delta_M B$ iff $A\delta B$. It is easy to see that (M, δ_M) is a basic proximity space.

We write $A \ll_\delta B$ (or simply $A \ll B$) if $A(-\delta)(X \setminus B)$, where "$-\delta$" means "not δ". When x is a point of X, we write $x \ll A$ in place of $\{x\} \ll A$.

A separated basic proximity δ on X which satisfies the condition

(EF) If $A \ll B$, then there exists a $C \subseteq X$ such that $A \ll C \ll B$ is called an *Efremovič proximity*; the pair (X, δ) is called *Efremovič proximity space.*

3.2 Let (X, δ) be a basic proximity. Then the operator cl_δ on $Exp(X)$ defined by

$$\mathrm{cl}_\delta(A) = \{x \in X : x\delta A\}.$$

is a Čech closure operator (see [5]). Hence $\tau_\delta = \{X \setminus A : A = \mathrm{cl}_\delta(A)\}$ is a topology on X. It is well known that the (Kuratowski) closure operator cl_{τ_δ} generated by τ_δ could not coincide with cl_δ. If δ is a *Lodato proximity* (i.e. a basic proximity such that $\mathrm{cl}_\delta(A)\delta\mathrm{cl}_\delta(B)$ implies $A\delta B$) then cl_δ coincides with cl_{τ_δ}; if δ satisfies the axiom (EF) then (X, τ_δ) is a completely regular space; if, moreover, δ is separated (i.e. δ is an Efremovič proximity) then (X, τ_δ) is a Tychonoff space ([33],[29]). Every Efremovič proximity is a Lodato proximity.

3.3 If (X, τ) is a topological space, we say that (X, τ) *admits a proximity*, if there is a basic proximity δ on X such that $\tau = \tau_\delta$; in this case we say also that δ is a *proximity on the space* (X, τ).

Examples 3.4 Here are few examples of proximity spaces:

1. Let X be a set having at least two points. For $A, B \subseteq X$, set $A\delta B \iff A \neq \emptyset$ and $B \neq \emptyset$. This is the *trivial basic proximity in* X. It is not separated. It satisfies the axiom (EF) but $(Exp(X), \delta)$ does not satisfy (C6).

2. Let (X, τ) be a T_4-space and define

   $$A\delta B \iff \mathrm{cl}(A) \cap \mathrm{cl}(B) \neq \emptyset.$$

   Then δ is an Efremovič proximity on the space X. We call it *standard proximity.*

   If (X, τ) is not a discrete space then $(Exp(X), \delta)$ does not satisfy the axiom (C6) (indeed, take a point x such that $\{x\}$ is not an open set; then there is no non-empty subset Y of X such that $Y \ll \{x\}$).

3. Let (X, τ) be a locally compact Hausdorff space, and, for $A, B \subseteq X$, define $A(-\delta)B$ iff $\mathrm{cl}(A) \cap \mathrm{cl}(B) = \emptyset$ and either $\mathrm{cl}(A)$ or $\mathrm{cl}(B)$ is compact. Then δ is an Efremovič proximity on the space X.

4. Let (X, d) be a metric space and, for $A, B \subseteq X$, define $A\delta B$ iff $d(A, B) = 0$, where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. Then δ is an Efremovič proximity on the space (X, τ_d).

5. Let (X, τ) be a completely regular space. Recall that two subsets A, B of X are *completely separated* iff there is a continuous real-valued function $f : X \longrightarrow [0,1]$ such that $f(A) = 0$ and $f(B) = 1$. We can define a basic proximity δ on X, satisfying the axiom (EF), by $A(-\delta)B$ iff A and B are completely separated.

Fact 3.5 *(see, e.g., [29]) If (X, τ) is a compact Hausdorff space, then it admits a unique Efremovič proximity δ, namely the standard one.*

Fact 3.6 (see, e.g., [29]) *Let (X, δ) be a Lodato proximity space and $A, B \subseteq X$. Then $A \ll_\delta B$ implies $\mathrm{cl}(A) \ll_\delta B$ and $A \ll_\delta \mathrm{Int}(B)$.*

Definition 3.7 Let X be a set. A *stack* on X is a family $\mathcal{S}$ of subsets of X satisfying the condition

$$B \supseteq A \in \mathcal{S} \Rightarrow B \in \mathcal{S}.$$

A *grill* ([6]) $\mathcal{G}$ on X is a stack on X satisfying $\emptyset \notin \mathcal{G}$ and

$$(A \cup B) \in \mathcal{G} \;\Rightarrow\; (A \in \mathcal{G} \text{ or } B \in \mathcal{G}).$$

Definition 3.8 Let (X, δ) be a basic proximity space. A grill $\mathcal{G}$ on X is called a *clan on* (X, δ) ([37]) iff

$$A, B \in \mathcal{G} \;\Rightarrow\; A\delta B.$$

A clan σ on (X, δ) is called a *cluster* ([25]) iff the following condition is satisfied:

(CL) If $A \subseteq X$ and $A\delta B$ for every $B \in \sigma$, then $A \in \sigma$.

For each $x \in X$, the collection $\sigma_x = \{A \subseteq X : A\delta x\}$ is a cluster. Such a cluster is called *point cluster.* If $\{x\} \in \sigma$ for some $x \in X$, then $\sigma = \sigma_x$. If (X, δ) is separated, then no cluster can contain more than one point.

Recall that a proximity space (X, δ) is said to be *compact* iff the topological space (X, τ_δ) is compact. We will need the following well-known result:

Theorem 3.9 *([25],[29]) An Efremovič proximity space is compact iff every cluster in the space is a point cluster.*

We say that a subset A of a topological space X is *regular closed* if $A = \mathrm{cl}(\mathrm{Int}(A))$. Clearly, A is regular closed iff it is a closure of an open set B.

Recall that for any topological space (X, τ), the collection $RC(X)$ of regular closed sets can be made into a complete Boolean algebra $(RC(X), 0, 1, \wedge, \vee, ^*)$, by setting

$$1 = X, 0 = \emptyset, A^* = \mathrm{cl}(X \setminus A), A \vee B = A \cup B, A \wedge B = (A^* \vee B^*)^* = \mathrm{cl}(\mathrm{Int}(A \cap B)).$$

The infinite operations are given by the formulas $\bigvee\{A_\gamma : \gamma \in \Gamma\} = \mathrm{cl}(\bigcup\{A_\gamma : \gamma \in \Gamma\})$, and $\bigwedge\{A_\gamma : \gamma \in \Gamma\} = \mathrm{cl}(\mathrm{Int}(\bigcap\{A_\gamma : \gamma \in \Gamma\}))$.

The following observation will be useful in the sequel:

Lemma 3.10 *Let (X, δ) be an Efremovič proximity space and $A, B \subseteq X$. If $A \ll_\delta B$ then there exists a regular closed set C such that $A \ll_\delta C \ll_\delta B$.*

Proof. By (EF), there exists a $D \subseteq X$ such that $A \ll D \ll B$. Applying 3.6, we obtain

$$A \ll_\delta \mathrm{Int}(D) \subseteq \mathrm{cl}(\mathrm{Int}(D)) \subseteq \mathrm{cl}(D) \ll_\delta B,$$

and thus, $A \ll_\delta \mathrm{cl}(\mathrm{Int}(D)) \ll_\delta B$. □

Next, we will consider connection algebras over proximity spaces.

Lemma 3.11 *Let (X, δ) be an Efremovič proximity space. Then, $(RC(X), \delta)$ is a connection algebra.*

Proof. The verification of axioms (C1) – (C4) is straightforward; (C5) follows from Lemma 3.10. For proving $(C6'')$ (which is equivalent to (C6)), let $A \in RC(X)$ and $A \neq \emptyset$. Then there exists a point $x \in \mathrm{Int}(A)$. Obviously, $x \ll \mathrm{Int}(A)$. Applying 3.10, we obtain that there exists $B \in RC(X)$ such that $x \ll B \ll A$. So, $B \neq \emptyset$ and $B \ll A$. □

Definition 3.12 Let (X, δ) be an Efremovič proximity space. $(RC(X), \delta)$ is called a *proximity connection algebra over* (X, δ). Following the definition in Example 2 of 3.4, we call it a *standard proximity connection algebra*, if

$$A \delta B \Longleftrightarrow A \cap B \neq \emptyset,$$

for all $A, B \in RC(X)$.

In the next section we will show that it suffices to consider only standard proximity connection algebras.

4 A representation theorem for connection algebras

In this section we shall prove that each connection algebra can be isomorphically embedded into a standard proximity connection algebra. Our strategy follows the proof of the Stone representation theorem for Boolean algebras. The points in a Stone space $\boldsymbol{S}(B)$ are the maximal filters in B. In connection algebras, the points of the representation space will be some analogues of maximal filters, called *clusters*. We take the notion of a cluster from the theory of proximity spaces, and our definition is just the lattice-theoretic translation of the corresponding definition of cluster (see Definition 3.8 here). Many statements about clusters in connection algebras have proofs which are identical (up to the aforementioned lattice-theoretical translation) to the proofs of the corresponding statements for clusters in proximity spaces. When such identical proofs exist we will refer to the corresponding statements and their proofs.

Throughout this Section we suppose that (B, C) is a connection algebra.

4.1 A non-empty subset Γ of B is called a *clan in* (B, C) if the following conditions are satisfied:

(K1) If $x, y \in \Gamma$ then xCy;

(K2) If $x < y$ and $x \in \Gamma$ then $y \in \Gamma$;

(K3) If $x \vee y \in \Gamma$ then $x \in \Gamma$ or $y \in \Gamma$.

A clan Γ in (B, C) is called a *cluster in* (B, C) if it satisfies the following condition:

(CLU) If xCy for every $y \in \Gamma$, then $x \in \Gamma$.

The set of all clusters in (B, C) is denoted by $\mathrm{Clust}(B, C)$ or simply by $\mathrm{Clust}(B)$. It is not hard to see that (the proofs are similar to those given in [37]):

(i) each clan is contained in a maximal clan;

(ii) each maximal clan is a cluster.

Recall that a subset F of B is called a *filter in* B if it satisfies the following conditions: (1) $0 \notin F$, (2) $a, b \in F$ implies that $a \wedge b \in F$, and (3) $a \leq b$ and $a \in F$ imply that $b \in F$. An *ultrafilter in* B is a maximal (with respect to the inclusion) filter in B.

We will now imitate Leader's proof of Smirnov Compactification Theorem (see [25], [29]). We will define a suitable proximity δ_B on $\mathrm{Clust}(B)$ and show that $(\mathrm{Clust}(B), \delta_B)$ is compact. Then, we will find an injective Boolean homomorphism h which preserves the connection relation C from B into the standard proximity connection algebra over $(\mathrm{Clust}(B), \delta_B)$.

The following properties of clusters will be helpful later:

Lemma 4.2 *Let* $\Gamma \in \mathrm{Clust}(B)$, *and* $a, b \in B$.

i. If aCb, *then there is some* $\Delta \in \mathrm{Clust}(B)$ *such that* $a \in \Delta$ *and* $b \in \Delta$.

ii. $a^* \in \Gamma$ *iff for all* $b, c \in B$, $c \in \Gamma$ *and* $b \vee a = 1$ *imply* cCb.

Proof.

i. The proof is analogous to the one of Theorem 5.14 of [29].

ii. If $a^*, c \in \Gamma$ and $a \vee b = 1$, then $a^* \leq b$. It follows from (K2) that $b \in \Gamma$, and hence, cCb (by (K1)).

Conversely, suppose that for all $b, c \in B$, $c \in \Gamma$ and $b \vee a = 1$ imply cCb. Setting $b = a^*$, we obtain that cCa^* for all $c \in \Gamma$, and thus, $a^* \in \Gamma$ (by (CLU)). □

Let us denote by PC_B the powerset of $\mathrm{Clust}(B)$. We define a function $h : B \to PC_B$ by

$$h(a) = \{\Gamma \in \mathrm{Clust}(B) : a \in \Gamma\} \tag{1}$$

in analogy to the Stone representation theorem for Boolean algebras. The following properties are easily proved from Lemma 4.2:

Lemma 4.3

i. $h(0) = \emptyset$, $h(a) = \mathrm{Clust}(B) \iff a = 1$.

ii. $h(a \vee b) = h(a) \cup h(b)$.

iii. aCb *iff* $h(a) \cap h(b) \neq \emptyset$.

Proof. i. Obviously, $h(0) = \emptyset$ and $h(1) = \mathrm{Clust}(B)$. Let $h(a) = \mathrm{Clust}(B)$. Then $a \in \Gamma$ for every $\Gamma \in \mathrm{Clust}(B)$. Suppose that $a \neq 1$. Then $1 \not\leq a$ and hence, by $(C6')$ (see 2.2), there exists an element b of B such that $bC1$ and $b(-C)a$. By Lemma 4.2(i), there exists a $\Gamma \in \mathrm{Clust}(B)$ such that $b \in \Gamma$. Since $a \in \Gamma$, we obtain that aCb and this is a contradiction. Therefore, $a = 1$.

ii. This follows immediately from (K3) and (K2) (see 4.1).

iii. If aCb, then Lemma 4.2(i) tells us that $h(a) \cap h(b) \neq \emptyset$. Conversely, if $h(a) \cap h(b) \neq \emptyset$, then aCb (by (K1)). □

4.4 Let (B, C) be a connection algebra. We set, for $X, Y \subseteq \mathrm{Clust}(B)$,

$$X\delta_B Y \text{ iff } (\forall x, y \in B)[(x \in \bigcap X \text{ and } y \in \bigcap Y) \text{ imply } xCy].$$

Using the definition of h, we obtain

$$X\delta_B Y \text{ iff } (\forall x, y \in B)[(X \subseteq h(x) \text{ and } Y \subseteq h(y)) \text{ imply } xCy].$$

The proof of the next result can be obtained by (part of) the proof of the Smirnov Compactification Theorem in [29] (Lemma 7.2):

Theorem 4.5 *Let* (B, C) *be a connection algebra. Then* $(\mathrm{Clust}(B), \delta_B)$ *is an Efremovič proximity space.*

4.6 Let $\mathcal{T}_{\mathrm{Clust}}$ be the topology on $\mathrm{Clust}(B)$ induced by δ_B. Then, for each $M \subseteq \mathrm{Clust}(B)$, we have

$$\mathrm{cl}(M) = \{\Gamma \in \mathrm{Clust}(B) : (\forall x, y \in B)[(x \in \Gamma \text{ and } M \subseteq h(y)) \Longrightarrow xCy]\}. \tag{2}$$

Theorem 4.7 *Let* (B, C) *be a connection algebra. Then* $(\mathrm{Clust}(B), \delta_B)$ *is compact.*

Proof. By 3.9, it is enough to show that every cluster in Clust(B) is a point cluster.

Let $\mathcal{C}$ be a cluster in $(\mathrm{Clust}(B), \delta_B)$. Put $\kappa = \{a \in B : h(a) \in \mathcal{C}\}$. We shall prove that κ is a clan in B. Note that $1 \in \kappa$.

Let $a \leq b$ and $a \in \kappa$. Then $h(a) \subseteq h(b)$, $h(a) \in \mathcal{C}$ and hence $h(b) \in \mathcal{C}$. Therefore, $b \in \kappa$.

If $a \vee b \in \kappa$ then $h(a \vee b) \in \mathcal{C}$, i.e. $h(a) \cup h(b) \in \mathcal{C}$. Hence $h(a) \in \mathcal{C}$ or $h(b) \in \mathcal{C}$, which means that $a \in \kappa$ or $b \in \kappa$.

Let $a, b \in \kappa$. Then $h(a), h(b) \in \mathcal{C}$ and hence $h(a)\delta_B h(b)$. This obviously implies (see the definition of δ_B) that aCb.

Therefore, κ is a clan in (B, C). Now, by 4.1, there exists a maximal clan ξ in (B, C) containing κ and $\xi \in \mathrm{Clust}(B)$. We shall show that $\{\xi\} \in \mathcal{C}$, i.e. that $\mathcal{C}$ is the point cluster σ_ξ in $(\mathrm{Clust}(B), \delta_B)$.

Let $\Gamma \in \mathcal{C}$. We shall prove that $\{\xi\}\delta_B\Gamma$, i.e. that if $x, y \in B$ are such that $\{\xi\} \subseteq h(x)$ and $\Gamma \subseteq h(y)$ then xCy. So, let x, y be as above. Then $\xi \in h(x)$, which means that $x \in \xi$. Hence xCa, for any $a \in B$ such that $h(a) \in \mathcal{C}$. Since $\Gamma \in \mathcal{C}$ and $\Gamma \subseteq h(y)$, we obtain that $h(y) \in \mathcal{C}$. Therefore, xCy.

We have proved that $\{\xi\}\delta_B\Gamma$ for every $\Gamma \in \mathcal{C}$. Thus $\{\xi\} \in \mathcal{C}$, i.e. $\mathcal{C} = \sigma_\xi$. $\square$

The following well-known variant of the famous "Grill Lemma", which can be proved exactly as Lemma 5.7 of [29], is valid for Boolean algebras:

Theorem 4.8 *Let $(B, 0, 1, \vee, \wedge, ^*)$ be a Boolean algebra and G be a subset of B such that $0 \notin G$ and $a \vee b \in G$ iff $a \in G$ or $b \in G$. If $a_0 \in G$ then there exists an ultrafilter U in B such that $a_0 \in U$ and $U \subseteq G$.*

The next theorem can be proved exactly as Theorem 5.8 of [29]:

Theorem 4.9 *A subset Γ of a connection algebra (B, C) is a cluster iff there exists an ultrafilter U in B such that*

$$\Gamma = \{a \in B : aCb \textit{ for every } b \in U\}. \tag{3}$$

Moreover, given Γ and $a_0 \in \Gamma$, there exists an ultrafilter U in B satisfying (3) which contains a_0.

Corollary 4.10 *Let (B, C) be a connection algebra and U be an ultrafilter in B. Then there exists a unique cluster Γ in (B, C) containing U.*

Finally, the following simple result can be proved exactly as Lemma 5.6 of [29]:

Fact 4.11 *Let (B, C) be a connection algebra, Γ_1 and Γ_2 be two clusters in (B, C). If $\Gamma_1 \subseteq \Gamma_2$, then $\Gamma_1 = \Gamma_2$.*

We are now ready to prove the representation theorem for connection algebras.

Theorem 4.12 (Representation Theorem for connection algebras)
(a) Each connection algebra (B, C) *can be embedded into a standard proximity connection algebra* $(RC(X), \delta)$*, where* (X, δ) *is a compact Efremovič proximity space. When* B *is complete this embedding becomes a complete isomorphism.*

(b) There exists a bijective correspondence between the class of all (up to isomorphism) complete connection algebras and the class of all (up to homeomorphism) compact Hausdorff spaces.

Proof. Let (B, C) be a connection algebra and $h : B \to PC_B$ be defined by (1). Our aim is to prove that h is an injective Boolean homomorphism $B \to RC(\mathrm{Clust}(B))$ such that $aCb \iff h(a) \cap h(b) \neq \emptyset$. This will show that (B, C) is isomorphic to a substructure of the proximity connection algebra over $(\mathrm{Clust}(B), \delta_B)$, which is a standard proximity connection algebra by 3.5.

We first show that $h(a^*) = \mathrm{cl}(-h(a))$ (we write "$-h(a)$" for "$\mathrm{Clust}(B) \setminus h(a)$"):

$$\begin{aligned}
\Gamma \in h(a^*) &\iff a^* \in \Gamma \\
&\iff (\forall b, c \in B)[(c \in \Gamma \text{ and } b \vee a = 1) \Longrightarrow cCb], \text{ by 4.2(ii)} \\
&\iff (\forall b, c \in B)[(c \in \Gamma \text{ and } h(b \vee a) = \mathrm{Clust}(B)) \Longrightarrow cCb], \text{ by 4.3(i)} \\
&\iff (\forall b, c \in B)[(c \in \Gamma \text{ and } -h(a) \subseteq h(b)) \Longrightarrow cCb], \text{ by 4.3(ii)} \\
&\iff \Gamma \in \mathrm{cl}(-h(a)).
\end{aligned}$$

This implies that h is well defined, i.e. that $h(a) \in RC(X)$, since

$$h(a) = \mathrm{cl}(-h(a^*)) = \mathrm{cl}(-\mathrm{cl}(-h(a))) = \mathrm{cl}(\mathrm{Int}(h(a))).$$

Furthermore, it shows that $h(a^*) = \mathrm{cl}(-h(a)) = h(a)^*$, so that h preserves complements. Together with Lemma 4.3, it follows that h is a Boolean homomorphism.

To show that h is injective, suppose that $a \neq b$, and let w.l.o.g. $a \not\leq b$. By $(C6')$, there is some $c \in B$ such that aCc and $b(-C)c$. Let $\Gamma \in \mathrm{Clust}(B)$ be such that $a, c \in \Gamma$ (see Lemma 4.2(i)). It follows now from $b(-C)c$ and (K1) that $b \notin \Gamma$.

By Lemma 4.3(iii), we have that, for $a, b \in B$, aCb iff $h(a) \cap h(b) \neq \emptyset$.

For proving the last statement of the first assertion of the theorem, we need some simple observations:

Fact 4.13 *Let* (B, C) *be a connection algebra. Then the family* $\{h(a) : a \in B\}$ *is a closed base of the topological space* $(\mathrm{Clust}(B), \mathcal{T}_{\mathrm{Clust}})$.

Proof. Let M be a closed subset of $(\mathrm{Clust}(B), \mathcal{T}_{\mathrm{Clust}})$. Hence $M = \mathrm{cl}(M)$. Let $\Gamma \notin M$. Then, by (2), there exist $x, y \in B$ such that $x \in \Gamma$, $M \subseteq h(y)$ and

$x(-C)y$. This implies that $\Gamma \notin h(y)$ (indeed, if $\Gamma \in h(y)$ then $y \in \Gamma$ and thus xCy, a contradiction). So, we have that $M \subseteq h(y)$ and $\Gamma \notin h(y)$. This means that $M = \bigcap\{h(y) : M \subseteq h(y)\}$. We have shown above that $h(y)$ is a closed set, for each $y \in B$. So, $\{h(a) : a \in B\}$ is a closed base of $(\mathrm{Clust}(B), \mathcal{T}_{\mathrm{Clust}})$. □

Corollary 4.14 *Let* (B, C) *be a connection algebra. Then the family* $\{\mathrm{Int}(h(a)) : a \in B\}$ *is an open base of the topological space* $(\mathrm{Clust}(B), \mathcal{T}_{\mathrm{Clust}})$.

Proof. Put $X = \mathrm{Clust}(B)$. By 4.13, we have that the family $\{X \setminus h(a) : a \in B\}$ is an open base of $(X, \mathcal{T}_{\mathrm{Clust}})$. If $a \in B$ then, as it was proved above, $h(a) = h((a^*)^*) = (h(a^*))^* = \mathrm{cl}(X \setminus h(a^*))$ and hence $X \setminus h(a) = X \setminus \mathrm{cl}(X \setminus h(a^*)) = \mathrm{Int}(h(a^*))$. So, the family $\{\mathrm{Int}(h(a^*)) : a \in B\} = \{\mathrm{Int}(h(a)) : a \in B\}$ is an open base of $(X, \mathcal{T}_{\mathrm{Clust}})$.
□

Fact 4.15 *Let* (B, C) *be a connection algebra and* $a, b \in B$. *Then* $h(a) \subseteq h(b)$ *implies* $a \leq b$.

Proof. This follows from the fact, established above, that h is an embedding. □

To finish the proof of the first assertion of our theorem, let B be a complete Boolean algebra and let $F \in RC(\mathrm{Clust}(B))$. Then, by 4.13, $F = \bigcap\{h(a) : F \subseteq h(a)\}$. Put $J = \{a \in B : F \subseteq h(a)\}$ and $j = \bigwedge J$. We will show that $F = h(j)$.

We have, by 4.14, that

$$\mathrm{Int}(F) = \bigcup\{\mathrm{Int}(h(b)) : \mathrm{Int}(h(b)) \subseteq \mathrm{Int}(F)\}. \tag{4}$$

Let $b \in B$ be such that $\mathrm{Int}(h(b)) \subseteq \mathrm{Int}(F)$. Then $\mathrm{Int}(h(b)) \subseteq h(a)$ for every $a \in J$ and hence $h(b) \subseteq h(a)$ for every $a \in J$. We obtain, by 4.15, that $b \leq a$, for every $a \in J$, so that $b \leq j$ and thus $h(b) \subseteq h(j)$. Now, (4) implies that $\mathrm{Int}(F) \subseteq h(j)$ and, hence, $F \subseteq h(j)$. On the other hand, we have that $h(j) \subseteq h(a)$, for every $a \in J$, and, therefore, $h(j) \subseteq \bigcap\{h(a) : a \in J\} = F$. So, $h(j) \subseteq F$ and $h(j) \supseteq F$, i.e. $h(j) = F$.

We have proved that $h : B \longrightarrow RC(\mathrm{Clust}(B), \mathcal{T}_{\mathrm{Clust}})$ is an isomorphism of the Boolean algebra B onto the Boolean algebra $RC(\mathrm{Clust}(B), \mathcal{T}_{\mathrm{Clust}})$ when B is complete. Hence h is a complete isomorphism (see, e.g. [32]).

(b) For every complete connection algebra (B, C), put

$$\Phi(B, C) = (\mathrm{Clust}(B), \mathcal{T}_{\mathrm{Clust}});$$

for every compact Hausdorff space (X, τ), put

$$\Psi(X, \tau) = (RC(X, \tau), C_X),$$

where, for every $A, B \in RC(X,\tau)$, $AC_X B$ iff $A \cap B \neq \emptyset$. Then, by (a), $h : (B, C) \longrightarrow \Psi(\Phi(B, C))$ is a complete isomorphism. Conversely, let (X, τ) be a compact Hausdorff space. We will prove that the space $(\mathrm{Clust}(RC(X)), \mathcal{T}_{\mathrm{Clust}})$ is homeomorphic to (X, τ). Let Γ be a cluster in $(RC(X), C_X)$. We will prove that there exists a unique point $x_\Gamma \in X$ such that $\Gamma = \sigma_{x_\Gamma}$, where $\sigma_x = \{A \in RC(X) : x \in A\}$ is the point cluster in $(RC(X), C_X)$.

By Theorem 4.9 there exists an ultrafilter $\mathcal{U}$ in $RC(X)$ such that

$$\Gamma = \{A \in RC(X) : A \cap B \neq \emptyset \text{ for every } B \in \mathcal{U}\}. \tag{5}$$

Since $A \wedge B \subseteq A \cap B$, for every $A, B \in RC(X)$, $\mathcal{U}$ is a family of closed subsets of the compact Hausdorff space (X, τ) having the finite intersection property. Hence $\bigcap\{B : B \in \mathcal{U}\} \neq \emptyset$ (see [16], 3.1.1). Let $x, y \in \bigcap\{B : B \in \mathcal{U}\}$ and $x \neq y$. There exist open neighborhoods Ox and Oy of x and y, respectively, such that $\mathrm{cl}(Ox) \cap \mathrm{cl}(Oy) = \emptyset$. Then $\mathrm{cl}(Ox), \mathrm{cl}(Oy) \in RC(X)$, $B \cap cl(Ox) \neq \emptyset$ for every $B \in \mathcal{U}$ and $B \cap cl(Oy) \neq \emptyset$ for every $B \in \mathcal{U}$. Now, (5) implies that $\mathrm{cl}(Ox), \mathrm{cl}(Oy) \in \Gamma$. Since Γ is a cluster, we have to have $\mathrm{cl}(Ox) C_X \mathrm{cl}(Oy)$, i.e. $\mathrm{cl}(Ox) \cap \mathrm{cl}(Oy) \neq \emptyset$, which is a contradiction. Thus $\bigcap\{B : B \in \mathcal{U}\}$ contains exactly one point, which will be denoted by x_Γ. Then we will have that $A \cap B \neq \emptyset$, for every $A \in \sigma_{x_\Gamma}$ and for every $B \in \mathcal{U}$. This implies, by (5), that $\sigma_{x_\Gamma} \subseteq \Gamma$. Using 4.11, we conclude that

$$\sigma_{x_\Gamma} = \Gamma. \tag{6}$$

The point x_Γ is the unique point x of X satisfying the equality $\sigma_x = \Gamma$ since, obviously, $\sigma_x \neq \sigma_y$ for $x \neq y$. So, a function $f : \mathrm{Clust}(RC(X)) \longrightarrow X$ is defined by the formula $f(\Gamma) = x_\Gamma$. Thanks to the equation (6), it is an injection. Since, for every $x \in X$, σ_x is a cluster in $(RC(X), C_X)$, we obtain that f is a bijection. For every $A \in RC(X)$, we have $f(h(A)) = f(\{\Gamma \in \mathrm{Clust}(RC(X)) : A \in \Gamma\}) = \{x_\Gamma : \Gamma \in \mathrm{Clust}(RC(X)), A \in \Gamma\} = \{x \in X : x \in A\} = A$. Since $RC(X)$ is a closed base of (X, τ) and, by 4.13, the family $\{h(A) : A \in RC(X)\}$ is a closed base of the space $(\mathrm{Clust}(RC(X)), \mathcal{T}_{\mathrm{Clust}})$, we conclude that $f : (\mathrm{Clust}(RC(X)), \mathcal{T}_{\mathrm{Clust}}) \longrightarrow (X, \tau)$ is a homeomorphism. So, Φ and Ψ are bijections. □

Corollary 4.16 (Isomorphism Theorem) *Each proximity connection algebra is isomorphic to a standard proximity connection algebra.*

Proof. Each proximity connection algebra is a connection algebra (see 3.11 and 3.12). Now all follows from Theorem 4.12. □

5 Local proximity spaces and local connection algebras

Definition 5.1 *([22])* A non-empty collection $\mathcal{B}$ of subsets of a set X is called a *boundedness in* X iff

(i) $A \in \mathcal{B}$ and $B \subseteq A$ implies $B \in \mathcal{B}$, and

(ii) $A, B \in \mathcal{B}$ implies $A \cup B \in \mathcal{B}$.

The elements of $\mathcal{B}$ are called *bounded sets.*

Definition 5.2 *([25])* A *local proximity space* $(X, \beta, \mathcal{B})$ consists of a set X, a basic proximity β on X, and a boundedness $\mathcal{B}$ in X subject to the following axioms:

(a) If $A \in \mathcal{B}$, $C \subseteq X$ and $A \ll C$ (where $\ll$ is defined with respect to β) then there exists a $B \in \mathcal{B}$ such that $A \ll B \ll C$;

(b) If $A\beta C$, then there is a $B \in \mathcal{B}$ such that $B \subseteq C$ and $A\beta B$.

When β is separated, then $(X, \beta, \mathcal{B})$ is said to be a *separated local proximity space.*

Note that (a) is equivalent to the following axiom:

(a') Let $A \subseteq X$ and $B \in \mathcal{B}$. If for every $C \in \mathcal{B}$ either $A\beta C$ or $(X \setminus C)\beta B$, then $A\beta B$.

Note also that (b) implies that every singleton set, and hence every finite subset of X, is bounded.

Two local proximity spaces $(X_1, \beta_1, \mathcal{B}_1)$ and $(X_2, \beta_2, \mathcal{B}_2)$ are said to be *isomorphic* if there exists a bijection between X_1 and X_2 which preserves in both directions the bounded sets and proximity relations.

Let (X, τ) be a topological space and $(X, \beta, \mathcal{B})$ be a local proximity space. We will say that $(X, \beta, \mathcal{B})$ *is a local proximity space on* (X, τ) iff $\tau_\beta = \tau$.

The next theorem of Leader ([25]) and its proof are of great importance for our investigations in this section.

Theorem 5.3 *([25]) Let (X, τ) be a Tychonoff space. Then there exists a bijection λ between the set of all (up to equivalence) locally compact Hausdorff extensions of (X, τ) and the set of all separated local proximity spaces on (X, τ). Namely, if Y is a locally compact Hausdorff extension of X then $\lambda(Y) = (X, \alpha_Y, \mathcal{B}_Y)$, where $\mathcal{B}_Y = \{F \subseteq X : \mathrm{cl}_Y(F)$ is compact in $Y\}$ and, for $A, B \subseteq X$, $A\alpha_Y B$ iff $\mathrm{cl}_Y(A) \cap \mathrm{cl}_Y(B) \neq \emptyset$ (here, for simplicity, we assume that $X \subseteq Y$).*

Definition 5.4 An algebraic system $B = (W, 0, 1, \vee, \wedge, ^*, \zeta, \mathbb{B})$ is called a *local connection algebra* (abbreviated as LCA), if $(W, 0, 1, \vee, \wedge, ^*)$ is a Boolean algebra, ζ is a binary relation on W and $\mathbb{B}$ is a subset of W, satisfying the following axioms:

(CC1) If $x \wedge y \neq 0$ then $x\zeta y$;

(CC2) If $x\zeta y$ then $x, y \neq 0$;

(CC3) $x\zeta y$ implies $y\zeta x$;

(CC4) $x\zeta(y \vee z)$ iff $x\zeta y$ or $x\zeta z$;

(BB1) $0 \in \mathbb{B}$;

(BB2) For $x, y \in W$, $x \leq y$ and $y \in \mathbb{B}$ implies $x \in \mathbb{B}$;

(BB3) $x, y \in \mathbb{B}$ implies $x \vee y \in \mathbb{B}$;

(BC1) If $x \in \mathbb{B}$, $z \in W$ and $x \ll z$ then there exists an $y \in \mathbb{B}$ such that $x \ll y \ll z$ (here " $\ll$", called *non-tangential inclusion*, is defined by "$x \ll y \Longleftrightarrow x(-\zeta)y^*$", where " $-\zeta$" means "non ζ"; sometimes we will write $\ll_\zeta$ instead of $\ll$);

(BC2) If $x\zeta y$ then there exists an element z of $\mathbb{B}$ such that $x\zeta(z \wedge y)$;

(BC3) If $x \neq 0$ then there exists an $y \in \mathbb{B} \setminus \{0\}$ such that $y \ll x$.

Usually, we shall write simply $(W, \zeta, \mathbb{B})$ for a local connection algebra. The elements of W will be called (spatial) *regions* and the relation ζ *local connection relation.*

We will say that two local connection algebras $(W, \zeta, \mathbb{B})$ and $(W_1, \zeta_1, \mathbb{B}_1)$ are *isomorphic* iff there exists a Boolean isomorphism $f : W \longrightarrow W_1$ such that, for $a, b \in W$, $a\zeta b$ iff $f(a)\zeta_1 f(b)$ and $f(a) \in \mathbb{B}_1$ iff $a \in \mathbb{B}$.

Remark 5.5 Note that if $(W, \zeta, \mathbb{B})$ is a local connection algebra and $1 \in \mathbb{B}$ then (W, ζ) is a connection algebra. Conversely, any connection algebra (B, C) can be regarded as the local connection algebra (B, C, B).

Example 5.6 Let $(X, \alpha, \mathcal{B})$ be a separated local proximity space. Then the triple $(RC(X), \alpha, \mathcal{B} \cap RC(X))$ is a local connection algebra.

Proof. It is clear that only the axioms (BC1)-(BC3) need to be checked. The first one follows immediately from 5.2(a) and the analogue of 3.10. Let us show that (BC3) is fulfilled. Take a non-empty regular closed set F. Then there exists a point $x \in \mathrm{Int}(F)$. Since $\{x\} \in \mathcal{B}$ and $x \ll F$, 5.2(a) implies that there is a $G \in \mathcal{B}$ such that $x \ll G \ll F$. As in 3.10, we can find a $G_1 \in RC(X)$ with $x \ll G_1 \ll G$. Then $G_1 \in \mathcal{B}$, $G_1 \neq \emptyset$ and $G_1 \ll F$, as required.

It remains to be shown that the axiom (BC2) is fulfilled. Let $A, B \in RC(X)$ and $B\alpha A$. By Leader Theorem 5.3, there exists a unique locally compact extension Y of X such that $X \subseteq Y$ and $\lambda(Y) = (X, \alpha, \mathcal{B})$. Then we have that $\mathrm{cl}_Y(A) \cap \mathrm{cl}_Y(B) \neq \emptyset$. Let $y \in \mathrm{cl}_Y(A) \cap \mathrm{cl}_Y(B)$. Since Y is locally compact, there exists an open in Y set U with compact closure such that $y \in U$. We have that $y \in \mathrm{cl}_Y(A) = \mathrm{cl}_Y(\mathrm{Int}_X(A))$ (because $A \in RC(X)$). Hence, setting $D = \mathrm{cl}_X(U \cap \mathrm{Int}_X(A))$, we obtain that $D \in RC(X)$, $\mathrm{cl}_Y(D) \subseteq \mathrm{cl}_Y(U)$ and hence $\mathrm{cl}_Y(D)$ is compact in Y, i.e. $D \in \mathcal{B}$. Further, we have $y \in U \cap \mathrm{cl}_Y(A) = U \cap \mathrm{cl}_Y(\mathrm{Int}_X(A)) \subseteq \mathrm{cl}_Y(U \cap \mathrm{cl}_Y(\mathrm{Int}_X(A))) = \mathrm{cl}_Y(U \cap \mathrm{Int}_X(A)) = \mathrm{cl}_Y(D)$, i.e. $\mathrm{cl}_Y(D) \cap \mathrm{cl}_Y(B) \neq \emptyset$. Therefore, $D\alpha B$, $D \in \mathcal{B} \cap RC(X)$ and $D \subseteq A$. This completes the proof. □

Notation 5.7 Let X be a topological space. We will denote by $Comp(X)$ the family of all compact subsets of X. We set also $CompReg(X) = Comp(X) \cap RC(X)$.

Corollary 5.8 *Let X be a locally compact Hausdorff space. Then the triple*

$$(RC(X), \alpha, CompReg(X)),$$

where, for $A, B \in RC(X)$, $A\alpha B$ iff $A \cap B \neq \emptyset$, is a local connection algebra.

Proof. By Theorem 5.3, $\lambda(X) = (X, \alpha_X, Comp(X))$, where, for $A, B \subseteq X$, $A\alpha_X B$ iff $\mathrm{cl}(A) \cap \mathrm{cl}(B) \neq \emptyset$, is a separated local proximity space. We can now use 5.6 for finishing the proof. $\square$

Definition 5.9 The local connection algebras $(RC(X), \alpha, CompReg(X))$, where X is a locally compact Hausdorff space (see 5.8) will be called *standard local connection algebras.*

We will need the following statement which should be a folklore; a particular case of it (when X is an open dense subset of Y) was proved by S. Koppelberg ([23], Lemma 4.3).

Lemma 5.10 *Let (X, τ) be a dense subspace of a topological space $(Y, \mathcal{T})$. Then the Boolean algebras $RC(X)$ and $RC(Y)$ are isomorphic.*

Proof. We shall prove that the function $f : RC(X) \longrightarrow RC(Y)$ defined by $f(A) = cl_Y A$, for every $A \in RC(X)$, is the desired isomorphism.

Let's show first that f is defined correctly. Take an $A \in RC(X)$. Then $A = \mathrm{cl}_X(\mathrm{Int}_X(A))$. We have to prove that $\mathrm{cl}_Y(A) = \mathrm{cl}_Y(\mathrm{Int}_Y(\mathrm{cl}_Y(A)))$. It is clear that $\mathrm{cl}_Y(A) \supseteq \mathrm{cl}_Y(\mathrm{Int}_Y(\mathrm{cl}_Y(A)))$. For proving the inclusion in the converse direction, it is enough to show that $\mathrm{Int}_X(A) \subseteq \mathrm{Int}_Y(\mathrm{cl}_Y(A))$, i.e. that $X \setminus \mathrm{cl}_X(X \setminus A) \subseteq X \cap (Y \setminus \mathrm{cl}_Y(Y \setminus \mathrm{cl}_Y(A)))$. Since X is dense in Y, we have that $\mathrm{cl}_Y(Y \setminus \mathrm{cl}_Y(A)) = \mathrm{cl}_Y(X \cap (Y \setminus \mathrm{cl}_Y(A)) = \mathrm{cl}_Y(X \setminus A)$. Hence $X \cap (Y \setminus \mathrm{cl}_Y(Y \setminus \mathrm{cl}_Y(A))) = X \setminus (X \cap \mathrm{cl}_Y(X \setminus A)) = X \setminus \mathrm{cl}_X(X \setminus A)$. Thus $f(A) \in RC(Y)$.

Define now another function $g : RC(Y) \longrightarrow RC(X)$ by the formula $g(B) = B \cap X$, where $B \in RC(Y)$.

For proving that g is defined correctly, let $B \in RC(Y)$. So $B = \mathrm{cl}_Y(\mathrm{Int}_Y(B))$. We have to show that $B \cap X = \mathrm{cl}_X(\mathrm{Int}_X(B \cap X))$. It is clear that $B \cap X \supseteq \mathrm{cl}_X(\mathrm{Int}_X(B \cap X))$. Conversely, let $x \in B \cap X$ and take an $U \in \tau$ such that $x \in U$. Then there exists $V \in \mathcal{T}$ such that $V \cap X = U$. We have that $V \cap \mathrm{Int}_Y(B) \neq \emptyset$. Since X is dense in Y, we obtain that $(V \cap \mathrm{Int}_Y(B)) \cap X \neq \emptyset$, i.e. $U \cap (X \cap \mathrm{Int}_Y(B)) \neq \emptyset$. But $X \cap \mathrm{Int}_Y(B) \subseteq \mathrm{Int}_X(B \cap X)$. Indeed, if $y \in X \cap \mathrm{Int}_Y(B)$ then there exists a $W \in \mathcal{T}$ such that $y \in W \subseteq B$; putting $W' = W \cap X$, we obtain that $y \in W' \subseteq B \cap X$

and $W' \in \tau$; thus $y \in \mathrm{Int}_X(B \cap X)$. So, we conclude that $U \cap \mathrm{Int}_X(B \cap X) \neq \emptyset$, which implies that $x \in \mathrm{cl}_X(\mathrm{Int}_X(B \cap X))$. Therefore $g(B) \in RC(X)$.

Further, we shall show that $g \circ f = id_{RC(X)}$ and $f \circ g = id_{RC(Y)}$.

Let $A \in RC(X)$. Then $g(f(A)) = X \cap \mathrm{cl}_Y(A) = A$.

Let now $B \in RC(Y)$. Then $f(g(B)) = \mathrm{cl}_Y(B \cap X)$. Obviously, $\mathrm{cl}_Y(B \cap X) \subseteq B$. Conversely, let $z \in B$. Let $O \in \mathcal{T}$ and $z \in O$. Since $B = \mathrm{cl}_Y(\mathrm{Int}_Y(B))$, we obtain that $O \cap \mathrm{Int}_Y(B) \neq \emptyset$. Thus $O \cap (X \cap \mathrm{Int}_Y(B)) \neq \emptyset$ and hence $O \cap (X \cap B) \neq \emptyset$. Therefore, $z \in \mathrm{cl}_Y(B \cap X)$. So, we have proved that $\mathrm{cl}_Y(B \cap X) = B$, i.e. $f(g(B)) = B$.

We conclude that $f : RC(X) \longrightarrow RC(Y)$ is a bijection. We will prove now that f preserves the Boolean operations. Indeed, we have that $f(\emptyset) = \emptyset$, $f(X) = Y$, $f(A_1 \vee A_2) = f(A_1 \cup A_2) = \mathrm{cl}_Y(A_1 \cup A_2) = \mathrm{cl}_Y(A_1) \cup \mathrm{cl}_Y(A_2) = f(A_1) \cup f(A_2) = f(A_1) \vee f(A_2)$. Finally, $f(A^*) = f(\mathrm{cl}_X(X \setminus A)) = \mathrm{cl}_Y(X \setminus A)$ and $(f(A))^* = (\mathrm{cl}_Y(A))^* = \mathrm{cl}_Y(Y \setminus \mathrm{cl}_Y(A)) = \mathrm{cl}_Y(X \cap (Y \setminus \mathrm{cl}_Y(A))) = \mathrm{cl}_Y(X \setminus A) = f(A^*)$. Therefore, f is an isomorphism between the Boolean algebras $(RC(X), \emptyset, X, \wedge, \vee, ^*)$ and $(RC(Y), \emptyset, Y, \wedge, \vee, ^*)$. This implies that F is a complete isomorphism (see, e.g., [32]). □

Proposition 5.11 *Let L be a locally compact Hausdorff space and X be a dense subspace of L. Then the local connection algebra $(RC(L), \alpha, CompReg(L))$ is isomorphic to the local connection algebra $(RC(X), \alpha_L, \mathcal{B}_L \cap RC(X))$ (see 5.8, 5.3 and 5.6 for the notations). In particular, if L is a compact Hausdorff space then $(RC(L), \alpha)$ and $(RC(X), \alpha_L)$ are isomorphic connection algebras.*

Proof. First of all we have, by 5.8, that $(RC(L), \alpha, CompReg(L))$ is a local connection algebra. Since L is a locally compact Hausdorff extension of X, the Leader Theorem 5.3 tells us that $\lambda(L) = (X, \alpha_L, \mathcal{B}_L)$ is a separated local proximity space. We obtain now, using 5.6, that $(RC(X), \alpha_L, \mathcal{B}_L \cap RC(X))$ is a local connection algebra. By Lemma 5.10, the function $f : RC(X) \longrightarrow RC(L)$, defined by $f(F) = \mathrm{cl}_L(F)$, is a Boolean isomorphism. Now, the definition of the family $\mathcal{B}_L$ (see 5.3) gives us that $f(F) \in CompReg(L)$ iff $F \in \mathcal{B}_L$. On the other hand, from the definition of α_L (see 5.3) we obtain immediately that, for $F, G \in RC(X)$, $F \alpha_L G$ iff $f(F) \alpha f(G)$. So, f is an isomorphism between the local connection algebras $(RC(X), \alpha_L, \mathcal{B}_L \cap RC(X))$ and $(RC(L), \alpha, CompReg(L))$.

When L is compact, we obviously have that $CompReg(L) = RC(L)$ and $X \in \mathcal{B}_L \cap RC(X)$, so that the last statement of our proposition follows from its first statement, 5.5 and 3.11. □

Theorem 5.12 *Let $(W, \zeta, \mathbb{B})$ be a local connection algebra and define a binary relation "C" on W by*

$$xCy \quad iff \quad x\zeta y \quad or \quad x, y \notin \mathbb{B}.$$

Then "C", called the Alexandroff extension of ζ, *is a connection relation on W and (W, C) is a connection algebra. (When it is necessary, we will write "C_ζ" instead of C.)*

Proof. The axioms $(C1) - (C3)$ (see 2.1) follow directly from the properties of ζ and $\mathbb{B}$ and the definition of C. Let us check that the axiom $(C4)$ is fulfilled.

Let $xC(y \vee z)$. Suppose first that $x\zeta(y \vee z)$. Then $x\zeta y$ or $x\zeta z$ and hence xCy or xCz. If $x \notin \mathbb{B}$ and $(y \vee z) \notin \mathbb{B}$ then $x \notin \mathbb{B}$ and (by (BB3) (see 5.4)) $y \notin \mathbb{B}$ or $z \notin \mathbb{B}$; hence xCy or xCz. Conversely, let xCy or xCz. If $x\zeta y$ or $x\zeta z$ then $x\zeta(y \vee z)$ and hence $xC(y \vee z)$. It remains to regard the case when $x \notin \mathbb{B}$, $y \notin \mathbb{B}$ and $z \notin \mathbb{B}$. Then $(y \vee z) \notin \mathbb{B}$ (by (BB3)) and hence $xC(y \vee z)$. So, the axiom $(C4)$ (see 2.1) is fulfilled.

We shall prove now that the axiom $(C5')$ (see 2.6) (which is equivalent to the axiom $(C5)$ (see 2.1)) is fulfilled. We shall write $x \ll_C y$ iff $x(-C)y^*$. Note that $x(-C)y$ iff $x(-\zeta)y$ and ($x \in \mathbb{B}$ or $y \in \mathbb{B}$). Hence, if $x \in \mathbb{B}$ (or $y^* \in \mathbb{B}$) and $x \ll_\zeta y$ then $x \ll_C y$.

Let $x \ll_C z$. Then $x(-C)z^*$ and hence $x(-\zeta)z^*$ and ($x \in \mathbb{B}$ or $z^* \in \mathbb{B}$). Therefore $x \ll_\zeta z$ and $z^* \ll_\zeta x^*$. If $x \in \mathbb{B}$ then, by (BC1) (see 5.4), there exists an $y \in \mathbb{B}$ such that $x \ll_\zeta y \ll_\zeta z$. Thus $x \ll_C y \ll_C z$. If $z^* \in \mathbb{B}$ then, by (BC1), there exists an element t of $\mathbb{B}$ such that $z^* \ll_\zeta t \ll_\zeta x^*$. This implies that $z^* \ll_C t \ll_C x^*$. So, $z^*(-C)t^*$ and $t(-C)x$, i.e. $x \ll_C t^* \ll_C z$. Therefore, the axiom $(C5)$ is fulfilled.

Let's, finally, verify that the axiom $(C6)$ (see 2.1) is fulfilled. Let $x \neq 1$. Then $x^* \neq 0$. By (BC3) (see 5.4), there exists an $y \in \mathbb{B} \setminus \{0\}$ such that $y \ll_\zeta x^*$. This implies that $y \ll_C x^*$, i.e. $y(-C)x$ and $y \neq 0$. So, (W, C) is a connection algebra. □

Lemma 5.13 *Let $(W, \zeta, \mathbb{B})$ be a local connection algebra and let $1 \notin \mathbb{B}$. Then $\sigma = \{x \in W : x \notin \mathbb{B}\}$ is a cluster in (W, C_ζ) (see Theorem 5.12 for the notation "C_ζ").*

Proof. By Theorem 5.12, we have that (W, C_ζ) is a connection algebra. We shall write simply "C" instead of "C_ζ".

Let $x, y \in \sigma$. Then $x \notin \mathbb{B}$ and $y \notin \mathbb{B}$. Hence, by the definition of C (see Theorem 5.12), we obtain that xCy. So the axiom (K1) (see 4.1) is fulfilled.

The axioms (K2) and (K3) (see 4.1) follow directly from (BB2) (see 5.4).

For showing that the axiom (CLU) (see 4.1) is also fulfilled, let xCy for every $y \in \sigma$. We will prove that $x \notin \mathbb{B}$ and then we will have that $x \in \sigma$. So, suppose that $x \in \mathbb{B}$. Then $x \neq 1$. Hence $x \ll_\zeta 1$. By (BC1) (see 5.4), there exists an element z of $\mathbb{B}$ such that $x \ll_\zeta z \ll_\zeta 1$. Then $x(-\zeta)z^*$. Since $1 \notin \mathbb{B}$, we obtain that $z^* \notin \mathbb{B}$. Thus $z^* \in \sigma$. This implies, by our assumption, that xCz^*. Therefore, by the definition of the relation "C" (see 5.12), we conclude that $x \notin \mathbb{B}$ (because we have that $x(-\zeta)z^*$), which is a contradiction.

So, σ is a cluster in (W, C). □

Theorem 5.14 (Representation Theorem for local connection algebras)

(a) Each local connection algebra $(W, \zeta, \mathbb{B})$ can be embedded into a standard local connection algebra $(RC(L), \alpha, CompReg(L))$, where L is a locally compact Hausdorff space. When W is complete this embedding becomes a complete isomorphism.

(b) (P. Roeper [31]) There exists a bijective correspondence between the class of all (up to isomorphism) complete local connection algebras and the class of all (up to homeomorphism) locally compact Hausdorff spaces.

Proof. (a) Let $(W, \zeta, \mathbb{B})$ be a local connection algebra. Let $C = C_\zeta$ be the Alexandroff extension of ζ (see 5.12). Then, by Theorem 5.12, (W, C) is a connection algebra. Put $X = \mathrm{Clust}(W, C)$. Let $\delta = \delta_W$ be the proximity on X defined as in 4.4. By Theorem 4.7, (X, δ) is a separated proximity space and, by Theorem 4.5, it is compact. If $1 \in \mathbb{B}$ then $\zeta = C$, (W, ζ) is a connection algebra and our assertion follows from Theorem 4.12. Let $1 \notin \mathbb{B}$. Then, by Lemma 5.13, the set $\sigma = \{x \in W : x \notin \mathbb{B}\}$ is a cluster in (W, C) and, hence, $\sigma \in X$. Denote by τ_C the topology on X defined by δ, put $L = L_{(W,\zeta,\mathbb{B})} = X \setminus \{\sigma\}$ and $\tau = \tau_{(W,\zeta,\mathbb{B})} = \tau_C|_L$. Then (L, τ) is a locally compact Hausdorff space. By Theorem 4.12, the function $h : (W, C) \longrightarrow RC(X, \tau_C)$, defined by $h(a) = \{\Gamma \in X : a \in \Gamma\}$, is an embedding of the connection algebra (W, C) into the standard connection algebra $RC(X, \tau_C)$; moreover, h becomes a complete isomorphism when the Boolean algebra W is complete. Put $H(a) = h(a) \cap L$, for each $a \in W$. We will show that:

(1) L is a dense subset of X;

(2) H is an embedding of the Boolean algebra W into the Boolean algebra $RC(L, \tau)$; H becomes a complete isomorphism when W is complete;

(3) $b \in \mathbb{B}$ iff $H(b) \in CompReg(L)$;

(4) $a\zeta b$ iff $H(a) \cap H(b) \neq \emptyset$.

In other words, H will be the desired embedding of the local connection algebra $(W, \zeta, \mathbb{B})$ into the standard local connection algebra $(RC(L, \tau), \alpha, CompReg(L))$ (where, for $A, B \in RC(L, \tau)$, $A\alpha B$ iff $A \cap B \neq \emptyset$) and H will become a complete isomorphism when W is a complete Boolean algebra.

To prove (1), recall that, by 4.13, $\{X \setminus h(a) : a \in W\}$ is an open base of (X, τ_C). As it follows from the definition of σ, $\sigma \in X \setminus h(a)$ iff $a \in \mathbb{B}$. So, let $a \in \mathbb{B}$. Since $1 \notin \mathbb{B}$, we have that $1 \not\leq a$. Then, by $(C6')$, there is an element z of W such that $zC1$ and $z(-C)a$. Thus $z \neq 0$ and, using (BC3), we can found an element b of $\mathbb{B} \setminus \{0\}$ such that $b \ll_\zeta z$. Then $b \leq z$ and hence, by 2.5(1), $b(-C)a$. Therefore,

$bC1$, $b \in \mathbb{B}$ and $b(-C)a$. Now, Lemma 4.2(i) implies that there is a cluster Γ in (W, C) such that $b \in \Gamma$; hence $\Gamma \neq \sigma$, i.e. $\Gamma \in L$. Since $b(-C)a$, we obtain that $a \notin \Gamma$. Therefore, $\Gamma \in L \cap (X \setminus h(a))$. This shows that L is a dense subset of X.

Let's prove (2). We have, by (1) above and by 5.10, that the function $g : RC(X, \tau_C) \longrightarrow RC(L, \tau)$, defined by $g(A) = A \cap L$ for every $A \in RC(X)$, is an isomorphism. Since $H = g \circ h$, we obtain that H is an embedding, which becomes a complete isomorphism when W is complete.

To establish (3), note that, for $a \in W$, $\sigma \in h(a) \leftrightarrow a \in \sigma \leftrightarrow a \notin \mathbb{B}$ and hence $h(a) \subseteq L \leftrightarrow a \in \mathbb{B}$; but for any $a \notin \mathbb{B}$, we have that $\sigma \in h(a) = \mathrm{cl}_X(H(a))$ (see 5.10); hence $H(a)$ is compact iff $h(a) \subseteq L$ iff $a \in \mathbb{B}$.

Finally, we will show that (4) takes place. Let $a, b \in W$ and $a\zeta b$. Then, by (BC2), there exist $a_1, b_1 \in \mathbb{B}$ such that $a_1 \leq a$, $b_1 \leq b$ and $a_1 \zeta b_1$. Then $a_1 C b_1$ and hence $h(a_1) \cap h(b_1) \neq \emptyset$. Since, by (3) above, $h(a_1) = H(a_1) \subseteq h(a) \cap L = H(a)$ and $h(b_1) = H(b_1) \subseteq h(b) \cap L = H(b)$, we obtain that $H(a) \cap H(b) \neq \emptyset$.

Let's prove the implication in the converse direction. Take $a, c \in W$ for which $H(a) \cap H(c) \neq \emptyset$. Then there exists a cluster $\Gamma \in h(a) \cap h(c) \cap L$. Since $\Gamma \neq \sigma$, there exists an element b_0 of $\mathbb{B}$ belonging to Γ (indeed, if all elements of Γ were unbounded, then $\Gamma \subseteq \sigma$, which would imply, by 4.11, that $\Gamma = \sigma$). Then $b_0 \neq 1$ and hence $b_0 \ll_\zeta 1$. By (BC1), there exists an element b of $\mathbb{B}$ such that $b_0 \ll_\zeta b \ll_\zeta 1$. Hence $b \in \Gamma$ and $b_0(-\zeta)b^*$. By the definition of C, we obtain that $b_0(-C)b^*$, i.e. $b^* \notin \Gamma$. So, $\Gamma \in \mathrm{Int}_X(h(b))$, where $b \in \mathbb{B}$ (indeed, $\mathrm{Int}_X(h(b)) = X \setminus \mathrm{cl}_X(X \setminus h(b)) = X \setminus (h(b))^* = X \setminus h(b^*)$). We will show now that $\Gamma \in h(a) \wedge h(b) = h(a \wedge b)$ and $\Gamma \in h(c) \wedge h(b) = h(c \wedge b)$. This will imply that $a \wedge b \in \Gamma$ and $c \wedge b \in \Gamma$ and hence $(a \wedge b)C(c \wedge b)$; since $a \wedge b, c \wedge b \in \mathbb{B}$, we will obtain that $(a \wedge b)\zeta(c \wedge b)$ and, therefore, $a\zeta c$. So, let's show that $\Gamma \in h(a) \wedge h(b) = \mathrm{cl}_X(\mathrm{Int}_X(h(a) \cap h(b)))$. Supposing that this is not the case, we can find an open neighborhood U of Γ such that $U \cap \mathrm{Int}_X(h(a) \cap h(b)) = \emptyset$. Put $V = U \cap \mathrm{Int}_X(h(b))$. Then V is also an open neighborhood of Γ and $V \cap \mathrm{Int}_X(h(a) \cap h(b)) = \emptyset$. Hence $V \subseteq \mathrm{cl}_X(X \setminus (h(a) \cap h(b))) = \mathrm{cl}_X(X \setminus h(a)) \cup \mathrm{cl}_X(X \setminus h(b))$. Since $V \cap \mathrm{cl}_X(X \setminus h(b)) = \emptyset$, we obtain that $V \subseteq \mathrm{cl}_X(X \setminus h(a))$. This is a contradiction because $\Gamma \in h(a) = \mathrm{cl}_X(\mathrm{Int}_X(h(a)))$. So, we have proved that $\Gamma \in h(a) \wedge h(b)$. Analogously, we obtain that $\Gamma \in h(c) \wedge h(b)$. This finishes the proof of the first part of our theorem.

(b) For every complete local connection algebra $(W, \zeta, \mathbb{B})$, put $\Phi'(W, \zeta, \mathbb{B}) = (L_{(W,\zeta,\mathbb{B})}, \tau_{(W,\zeta,\mathbb{B})})$ (see (a) above for the notations); for every locally compact Hausdorff space (L, τ), put $\Psi'(L, \tau) = (RC(L, \tau), \zeta_L, CompReg(L, \tau))$, where, for every $A, B \in RC(L, \tau)$, $A\zeta_L B$ iff $A \cap B \neq \emptyset$. Then, by (a), $H : (W, \zeta, \mathbb{B}) \longrightarrow \Psi'(\Phi'(W, \zeta, \mathbb{B}))$ is a complete isomorphism. Conversely, let (L, τ) be a locally compact Hausdorff space. We will prove that the space $L_{(RC(L,\tau),\zeta_L,CompReg(L,\tau))}$, defined in (a) above, is homeomorphic to (L, τ). Let Γ be a bounded cluster in $(RC(L), C_{\zeta_L})$, i.e. $\Gamma \cap CompReg(L) \neq \emptyset$. We can prove exactly as in the proof of the part (b) of Theorem 4.12 that there exists a unique point $x_\Gamma \in L$ such that $\Gamma = \sigma_{x_\Gamma}$, where $\sigma_x = \{A \in RC(L) : x \in A\}$ is the point cluster in $(RC(L), \zeta_L, CompReg(L))$. Then a function $f' : L_{(RC(L,\tau),\zeta_L,CompReg(L,\tau))} \longrightarrow X$ can be defined by the formula $f'(\Gamma) = x_\Gamma$. Now, a proof completely analogous to the one of the part (b) of Theorem 4.12 shows that f' is a homeomorphism (note that now $\{H(A) : A \in RC(L)\}$ is a closed base of $L_{(RC(L,\tau),\zeta_L,CompReg(L,\tau))}$ and that every point cluster is a bounded cluster since the space L is locally compact). So, Φ' and Ψ' are bijections. □

Remark 5.15 Let $(W, \zeta, \mathbb{B})$ be a local connection algebra. Then, by Theorem 5.14, there exists a unique (up to homeomorphism) locally compact Hausdorff space L such that $(W, \zeta, \mathbb{B})$ is isomorphic to the standard local connection algebra

$$(RC(L), \alpha, CompReg(L)).$$

There are, however, many proximity-type models of $(W, \zeta, \mathbb{B})$. For instance, according to Proposition 5.11, all local connection algebras of the form $(RC(X), \alpha_L, \mathcal{B}_L \cap RC(X))$ (see 5.8, 5.3 and 5.6 for the notations) are models, where X is any dense subspace of L. One of our aims in this paper was to call the reader's attention to such models.

An analogous remark is valid for connection algebras: by Theorem 4.12, for any connection algebra (B, C), there exists a unique (up to homeomorphism) compact Hausdorff space K such that (B, C) is isomorphic to the standard connection algebra $(RC(K), \alpha)$ (i.e. for any $F, G \in RC(K)$, $F \alpha G$ iff $F \cap G \neq \emptyset$), but, by Proposition 5.11, for any dense subspace X of K, we have that (B, C) is isomorphic to the connection algebra $(RC(X), \alpha_K)$ (see 5.3 for the notations and note that a separated local proximity space $(X, \alpha_K, \mathcal{B}_K)$ becomes an Efremovič proximity space when $X \in \mathcal{B}_K$).

6 MVD-algebras

As we have already mentioned above, connection algebras could be equivalently defined as a pair of a Boolean algebra $B = (B, 0, 1, \vee, \wedge, ^*)$ and a binary relation $\ll$ subject to the following axioms:

($\ll$1) $x \ll y$ implies $x \leq y$;
($\ll$2) $0 \ll 0$;
($\ll$3) $x \leq y \ll z \leq t$ implies $x \ll t$;
($\ll$4) $x \ll z$ and $y \ll z$ implies $x \vee y \ll z$;
($\ll$5) If $x \ll z$ then $x \ll y \ll z$ for some $y \in B$;
($\ll$6) If $x \neq 0$ then there exists $y \neq 0$ such that $y \ll x$;
($\ll$7) $x \ll y$ implies $y^* \ll x^*$.

The proof of the equivalence of the two definitions of connection algebras is straightforward and analogous to the corresponding statement for proximity spaces (see Theorems 3.9 and 3.11 in [29]). One has just to show that xCy iff $x \not\ll y^*$.

In [28] Mormann introduces the notion of *enriched Boolean algebra* as a pair of a Boolean algebra $(B, \leq)$ and a binary relation $\ll$ (called by him *interior parthood*) for which ($\ll$1)-($\ll$6) hold and the axiom

($\ll 4^*$) $x \ll y$ and $x \ll z$ imply $x \ll y \wedge z$

is fulfilled.

To be precise, he writes $0 \ll x$ instead of ($\ll$2) and substitutes ($\ll$5) and ($\ll$6) with the following axiom

($\ll$5-6) $x \ll z$ and $x \neq z$ together imply $x \ll y \ll z$ for some $y \neq x$,

but, obviously, our expression of the axioms of enriched Boolean algebras is equivalent to that given by Mormann (indeed, let $x \ll z$ and $x \neq z$; then $z \wedge x^* \neq 0$ and, by ($\ll$6), there exists a $t \neq 0$ such that $t \ll (z \wedge x^*)$; hence, by ($\ll$3) and ($\ll$1), $t \ll z$ and $t \leq x^*$; by ($\ll$5), there exists an $u \in B$ with $x \ll u \ll z$; setting $y = u \vee t$, we obtain, using ($\ll$3) and ($\ll$4), that $x \ll y \ll z$; obviously $x \neq y$; so, ($\ll$5-6) implies ($\ll$5) and ($\ll$6); the converse implication is obvious). Thus, the difference between our connection algebras and Mormann's enriched Boolean algebras is that our axiom ($\ll$7) is substituted by the weaker axiom ($\ll 4^*$) (in the sense that having ($\ll$7) one can derive ($\ll 4^*$) from ($\ll 4$)).

Note that the extensionality axiom is fulfilled in the enriched Boolean algebras. Indeed, it is enough to show that $(x \leq y) \iff [\forall z : (z \ll x) \Rightarrow (z \ll y)]$. In the direction ($\Longrightarrow$), this follows from ($\ll$3). For proving the direction ($\Longleftarrow$), suppose that $x \not\leq y$. Then $z' = x \wedge y^* \neq 0$. Hence, by ($\ll$6), there exists an element $z \neq 0$ such that $z \ll z'$. Then, by ($\ll$3), $z \ll x$ and $z \ll y^*$. But $z \ll x$ implies that $z \ll y$. Thus we obtain, by ($\ll 4^*$), that $z \ll (y \wedge y^*)$, i.e., by ($\ll$1), $z = 0$, which is a contradiction.

In [28] Mormann affirms that for any enriched complete Boolean algebra $(B, \leq, \ll)$ there exists a locally compact Hausdorff space L such that $(B, \leq, \ll)$ is isomorphic to $(RO(L), \subseteq, \ll_L)$, where $RO(L)$ is the complete Boolean algebra of regular open subsets of L and, for any $U, V \in RO(L)$, $U \ll_L V$ iff $\mathrm{cl}(U)$ is compact and $\mathrm{cl}(U) \subseteq V$; conversely, for any locally compact Hausdorff space L, the triple $(RO(L), \subseteq, \ll_L)$ is an enriched complete Boolean algebra. Since the map $\nu : RO(L) \longrightarrow RC(L)$, defined by $\nu(U) = \mathrm{cl}(U)$, is an isomorphism between the complete Boolean algebras $RO(L)$ and $RC(L)$, and, for $U, V \in RO(L)$, $U \ll_L V$ iff $\nu(U) \subseteq \mathrm{Int}(\nu(V))$ and $\nu(U)$ is compact, we can say that $(B, \leq, \ll)$ is isomorphic to the enriched Boolean algebra $(RC(L), \subseteq, \ll_L)$, where, for all $F, G \in RC(L)$, $F \ll_L G$ iff F is compact and $F \subseteq \mathrm{Int}(G)$ (we hope that the use of the same notation ($\ll_L$) with different meanings will cause no confusion). Trying to prove this Mormann's representation theorem using proximity approach, we arrived to the following notion:

Definition 6.1 A triple $M = (B, \leq, \ll)$ is called an *MVD-algebra* if it is an enriched Boolean algebra and satisfies the following axiom

(*) If $x \ll 1$ then $y^* \ll x^*$ implies $x \ll y$.

When $(B, \leq)$ is a complete Boolean algebra, we will say that M is a *complete MVD-algebra.*

It follows immediately from the corresponding definitions that connection algebras coincide with MVD-algebras satisfying the additional axiom

$(\ll 2')$ $1 \ll 1$.

We will show below that the notion of MVD-algebra is equivalent to the notion of local connection algebra and hence we will obtain a representation theorem for MVD-algebras (see Theorem 6.5 below) using Theorem 5.14. This representation theorem sounds exactly as Mormann's representation theorem [28], gives the same semantics, so that it should imply that complete MVD-algebras and enriched complete Boolean algebras are equivalent notions. This is, however, not the case, as the next simple example shows; hence, Mormann's representation theorem [28] is not true.

Example 6.2 *There exist enriched complete Boolean algebras which are not MVD-algebras.*

Proof. Let X be a non-empty and non-discrete T_1-space. We define a relation " $\ll$ " on $Exp(X)$ by $A \ll B$ iff $cl(A) \subseteq B$. Then it is easy to verify that $(Exp(X), \subseteq, \ll)$ is an enriched complete Boolean algebra. We shall show that $(Exp(X), \subseteq, \ll)$ is not an MVD-algebra, i.e. it does not satisfy the axiom (*) (from 6.1). Indeed, let x be a non-isolated point of X. Put $B = A = X \setminus \{x\}$. Then, obviously, $A \ll X$ and $X \setminus B \ll X \setminus A$ (since $X \setminus B = \{x\}$ is a closed subset of X) but $A \not\ll B$, because A is not closed in X. □

Theorem 6.3 *The notions of local connection algebra and MVD-algebra are equivalent.*

Proof. Denote by $\mathcal{LCA}$ the class of all local connection algebras and by $\mathcal{MA}$ the class of all MVD-algebras. We shall define two functions $f : \mathcal{LCA} \longrightarrow \mathcal{MA}$ and $g : \mathcal{MA} \longrightarrow \mathcal{LCA}$ and we will show that $f \circ g = id_{\mathcal{MA}}$ and $g \circ f = id_{\mathcal{LCA}}$.

Let $(W, \zeta, \mathbb{B})$ be a local connection algebra.Then $x \leq y$ iff $x \wedge y = x$. We have that $x \ll_\zeta y$ iff $x(-\zeta)y^*$. Put $x \ll_M y$ iff $x \in \mathbb{B}$ and $x \ll_\zeta y$. We will prove that $(W, \leq, \ll_M)$ is a MVD-algebra and we will set $f((W, \zeta, \mathbb{B})) = (W, \leq, \ll_M)$.

For proving $(\ll 1)$, let $x \ll_M y$; then $x(-\zeta)y^*$ and hence $x \wedge y^* = 0$; this implies that $x \leq y$.

Since $0 \in \mathbb{B}$ and $0 \ll_\zeta 0$, we obtain that $0 \ll_M 0$, i.e. the axiom ($\ll$2) is fulfilled.

Let's verify the axiom ($\ll$3). Let $x \leq y \ll_M z \leq t$. Then $y \in \mathbb{B}$, $y(-\zeta)z^*$ and $x \vee y = y$. This implies, by (BB2) and (CC4), that $x \in \mathbb{B}$ and $x(-\zeta)z^*$. Since $z \leq t$, we have that $z^* \vee t^* = z^*$ and hence, by (CC4), $x(-\zeta)t^*$. So, $x \in \mathbb{B}$ and $x \ll_\zeta t$. Therefore, $x \ll_M t$.

For checking ($\ll$4), let $x \ll_M z$ and $y \ll_M z$. Then $x, y \in \mathbb{B}$, $x(-\zeta)z^*$ and $y(-\zeta)z^*$. Hence, by (BB3) and (CC4), $x \vee y \in \mathbb{B}$ and $(x \vee y)(-\zeta)z^*$. Therefore, $x \vee y \ll_M z$.

Let's prove that ($\ll$5) is fulfilled. Let $x \ll_M z$. Then $x \in \mathbb{B}$ and $x \ll_\zeta z$. By (BC1), there exists an $y \in \mathbb{B}$ such that $x \ll_\zeta y \ll_\zeta z$. This means that $x \ll_M y \ll_M z$.

For verifying ($\ll$6), let $x \neq 0$. Then, by (BC3), there exists an $y \in \mathbb{B} \setminus \{0\}$ such that $y \ll_\zeta x$. Therefore $y \ll_M x$ and $y \neq 0$.

Let's show that ($\ll 4^*$) is fulfilled. Let $x \ll_M y$ and $x \ll_M z$. Then $x \in \mathbb{B}$, $x(-\zeta)y^*$ and $x(-\zeta)z^*$. By (CC4), we obtain that $x(-\zeta)(y^* \vee z^*)$, i.e. $x(-\zeta)(y \wedge z)^*$. Hence $x \ll_M y \wedge z$.

For proving that (*) is fulfilled, let $x \ll_M 1$ and $y^* \ll_M x^*$. Then $x \in \mathbb{B}$ and $x(-\zeta)y^*$. Thus $x \ll_M y$.

So, $(W, \leq, \ll_M)$ is a MVD-algebra.

Let now $(B, \leq, \ll)$ be a MVD-algebra. Put

$$g((B, \leq, \ll)) = (B, \zeta_M, \mathbb{B}_M),$$

where $\mathbb{B}_M = \{x \in B : x \ll 1\}$ and, for $x, y \in B$,

$$x\zeta_M y \text{ iff there exists } z \in \mathbb{B}_M \text{ such that } (z \wedge x) \not\ll (z \wedge y)^*.$$

We shall prove that $(B, \zeta_M, \mathbb{B}_M)$ is a local connection algebra, i.e. that the definition of the function g is correct. In the proof of this claim, we will put, for short, $\mathbb{B} = \mathbb{B}_M$ and $\zeta = \zeta_M$.

First of all, we will note that, for $x, y \in B$,

$$x\zeta y \text{ iff there exists } z \in \mathbb{B} \text{ such that } (z \wedge x) \not\ll (z \wedge y)^* \text{ and } (z \wedge y) \not\ll (z \wedge x)^*.$$

Indeed, if $x\zeta y$ then there exists $z \in \mathbb{B}$ such that $(z \wedge x) \not\ll (z \wedge y)^*$. Since $z \wedge y \in \mathbb{B}$, (*) implies that $(z \wedge y) \not\ll (z \wedge x)^*$. The converse is clear.

¿From ($\ll$2) and ($\ll$3) we obtain that $0 \in \mathbb{B}$, i.e. the axiom (BB1) is fulfilled. ($\ll$3) implies that (BB2) is fulfilled and, by ($\ll$4), (BB3) is also fulfilled.

For verifying (CC1), let $x \wedge y \neq 0$. Then ($\ll$6) implies that there exists an element z of $B \setminus \{0\}$ such that $z \ll (x \wedge y)$. Hence, by ($\ll$3) and ($\ll$1), $z \in \mathbb{B}$, $z \wedge x = z$, $z \wedge y = z$ and $z \not\ll z^*$. This means that $x\zeta y$.

Obviously, ($\ll$2) and ($\ll$3) imply that $0 \ll x$ for any $x \in B$. This is what we need to establish the validity of (CC2). Indeed, let $x\zeta y$. Then there exists an element z of $\mathbb{B}$ such that $(z \wedge x) \not\ll (z \wedge y)^*$ and $(z \wedge y) \not\ll (z \wedge x)^*$. So, supposing that $x = 0$ or $y = 0$ we would obtain that $0 \not\ll (z \wedge y)^*$ or $0 \not\ll (z \wedge x)^*$, which is a contradiction. Hence, $x, y \neq 0$.

It is clear that (CC3) follows directly from the observation which we have made after the definition of the relation ζ. Let's show that (CC4) takes place.

Let $x\zeta(y \vee z)$. Then there exists an element t of $\mathbb{B}$ such that $(t \wedge x) \not\ll (t \wedge (y \vee z))^*$. Using ($\ll 4^*$), we obtain that

$$(t \wedge x) \not\ll (t \wedge y)^* \text{ or } (t \wedge x) \not\ll (t \wedge z)^*$$

If $(t \wedge x) \not\ll (t \wedge y)^*$ then $x\zeta y$. If $(t \wedge x) \not\ll (t \wedge z)^*$ then $x\zeta z$. So, $x\zeta(y \vee z)$ implies that $x\zeta y$ or $x\zeta z$.

Let now $x\zeta y$ or $x\zeta z$. Suppose, for example, that $x\zeta y$. Then there exists an $u \in \mathbb{B}$ such that $(u \wedge x) \not\ll (u \wedge y)^*$. Using ($\ll$3), we obtain immediately that $(u \wedge x) \not\ll (u \wedge y)^* \wedge (u \wedge z)^*$, i.e. $(u \wedge x) \not\ll (u \wedge (y \vee z))^*$. Hence we obtain that $x\zeta(y \vee z)$. When $x\zeta z$, the proof is analogous. Therefore, we have shown that the axiom (CC4) is fulfilled.

Let's verify that the axiom (BC1) is fulfilled. Let $x \in \mathbb{B}$, $z \in B$ and $x \ll_\zeta z$ (i.e. $x(-\zeta)z^*$). Since $x \ll 1$, there exists (by ($\ll$5)) an $u \in B$ such that $x \ll u \ll 1$. Then $u \in \mathbb{B}$. By the definition of the relation ζ, we have

$$x(-\zeta)z^* \text{ iff (for every } t \in \mathbb{B})(t \wedge x \ll (t \wedge z^*)^*).$$

Putting $t = u$, we obtain (since, by ($\ll$1), $x \leq u$) that $x \ll (u \wedge z^*)^*$, i.e. $x \ll (u^* \vee z)$. Since $x \ll u$, ($\ll 4^*$) implies that $x \ll (u \wedge z)$. Using again ($\ll$5), we find an $y \in B$ such that $x \ll y \ll u \wedge z$. Then $y \in \mathbb{B}$ and $x \ll y \ll z$. Noting that $a \ll b$ implies $a \ll_\zeta b$ (indeed, for every $t \in \mathbb{B}$, one has $t \wedge a \leq a \ll b \leq t^* \vee b = (t \wedge b^*)^*$ and hence $t \wedge a \ll (t \wedge b^*)^*$, which implies that $a(-\zeta)b^*$, i.e. $a \ll_\zeta b$), we conclude that $x \ll_\zeta y \ll_\zeta z$.

We shall verify now the axiom (BC2). Let $x\zeta y$. Then, by the definition of the relation ζ, there exists an element z of $\mathbb{B}$ such that $(z \wedge x) \not\ll (z \wedge y)^*$. This implies directly that $x\zeta(z \wedge y)$ (take just the same z).

Finally, (BC3) follows immediately from ($\ll$6) and the fact, proved above, that $y \ll x$ implies $y \ll_\zeta x$. Therefore, $(B, \zeta, \mathbb{B})$ is indeed a local connection algebra.

Let us show now that $g \circ f = id_{\mathcal{LCA}}$. Take a local connection algebra $(W, \zeta, \mathbb{B})$. Then $f((W, \zeta, \mathbb{B})) = (W, \leq, \ll_M)$. Let $g((W, \leq, \ll_M)) = (W, \zeta_M, \mathbb{B}_M)$ (see the corresponding definitions above). We will show that $\mathbb{B}_M = \mathbb{B}$ and $\zeta_M = \zeta$. We have, by our definitions, $x \ll_M y$ iff $x \in \mathbb{B}$ and $x \ll_\zeta y$ (where, as usual, $x \ll_\zeta y$ means that $x(-\zeta)y^*$); further, we have $\mathbb{B}_M = \{x \in W : x \ll_M 1\}$ and $x\zeta_M y$ iff there exists an element z of $\mathbb{B}_M$ such that $(z \wedge x) \not\ll_M (z \wedge y)^*$.

If $x \in \mathbb{B}_M$ then $x \ll_M 1$ and hence, by the definition of $\ll_M$, $x \in \mathbb{B}$. Thus $\mathbb{B}_M \subseteq \mathbb{B}$. Conversely, if $x \in \mathbb{B}$ then $x \ll_\zeta 1$ (because $x(-\zeta)0$) and hence, by the definition of $\ll_M$, $x \ll_M 1$, i.e. $x \in \mathbb{B}_M$. Therefore, $\mathbb{B} \subseteq \mathbb{B}_M$. So, $\mathbb{B}_M = \mathbb{B}$.

Let $x\zeta y$. By (BC2) and (CC3), there exist $u, v \in \mathbb{B}$ such that $u \leq x$, $v \leq y$ and $u\zeta v$. Then, by (BB3), $z = u \vee v \in \mathbb{B}$. Since $z \wedge x \geq u$ and $z \wedge y \geq y$, we obtain, by (CC4), that $(z\wedge x)\zeta(z\wedge y)$. Thus $(z\wedge x) \not\ll_\zeta (z\wedge y)^*$ and $(z\wedge y) \not\ll_\zeta (z\wedge x)^*$. Since $z \wedge x, z \wedge y \in \mathbb{B}$ (by (BB2)) and $\mathbb{B} = \mathbb{B}_M$ (as we have proved above), we obtain, by the definition of $\ll_M$, that $(z \wedge x) \not\ll_M (z \wedge y)^*$. According to the definition of the relation ζ_M, this implies that $x\zeta_M y$.

Let now $x\zeta_M y$. Then there exists an element z of $\mathbb{B}_M$ such that $(z \wedge x) \not\ll_M (z \wedge y)^*$. Since $z \wedge x, z \wedge y \in \mathbb{B}_M$ and $\mathbb{B}_M = \mathbb{B}$, we obtain, by the definition of the relation $\ll_M$, that $(z \wedge x) \not\ll_\zeta (z \wedge y)^*$, i.e. $(z \wedge x)\zeta(z \wedge y)$ and hence $x\zeta y$. We have proved that $\zeta = \zeta_M$. So, $g \circ f = id_{\mathcal{LCA}}$.

We will show, finally, that $f \circ g = id_{\mathcal{MA}}$. Let $(W, \leq, \ll)$ be a MVD-algebra, $g((W, \leq, \ll)) = (W, \zeta, \mathbb{B})$ and $f((W, \zeta, \mathbb{B})) = (W, \leq, \ll_M)$ (see the corresponding definitions above). We have to prove only that $\ll = \ll_M$. We have, by our definitions, that $\mathbb{B} = \{x \in W : x \ll 1\}$, $x\zeta y$ iff there exists an element z of $\mathbb{B}$ such that $(z \wedge x) \not\ll (z \wedge y)^*$; further, we have $x \ll_M y$ iff $x \in \mathbb{B}$ and $x \ll_\zeta y$ (where, as usual, $x \ll_\zeta y$ means that $x(-\zeta)y^*$). Obviously, $x(-\zeta)y$ iff (for every $z \in \mathbb{B}$) $[(z \wedge x) \ll (z \wedge y)^*]$ Thus we obtain that $x \ll_\zeta y$ iff $x(-\zeta)y^*$ iff (for every $z \in \mathbb{B}$) $[(z \wedge x) \ll (z \wedge y^*)^*]$ iff (for every $z \in \mathbb{B}$) $[(z \wedge x) \ll (z^* \vee y)]$ So,

$$x \ll_\zeta y \leftrightarrow (\forall z \in \mathbb{B})[(z \wedge x) \ll (z^* \vee y). \tag{7}$$

Let now $x \ll y$. Then $x \ll 1$ and hence $x \in \mathbb{B}$. For every $z \in \mathbb{B}$, we have $z \wedge x \leq x \ll y \leq y \vee z^*$, so that, by ($\ll$3), $z \wedge x \ll z^* \vee y$. Hence, (7) implies that $x \ll_\zeta y$. Since $x \in \mathbb{B}$, we conclude that $x \ll_M y$.

Conversely, let $x \ll_M y$. Then, by the definition of the relation $\ll_M$, $x \in \mathbb{B}$ and $x \ll_\zeta y$. We have to prove that $x \ll y$.

If $x = 1$ then, by ($\ll$1), $y = 1$. Since $x \in \mathbb{B}$, we obtain, by the definition of $\mathbb{B}$, that $x \ll 1$, i.e. $1 \ll 1$. Thus $x \ll y$.

Let $x \neq 1$. Since $x \ll 1$, ($\ll$5-6) (which is equivalent to ($\ll$5) and ($\ll$6), as we have proved above) implies that there exists a $z \neq x$ such that $x \ll z \ll 1$. Then $z \in \mathbb{B}$ and $x \leq z$, so that $z \wedge x = x$. Now, (7) implies (since $z \in \mathbb{B}$ and $x \ll_\zeta y$) that $x \ll (z^* \vee y)$. Since $x \ll z$, $(\ll 4^*)$ implies that $x \ll (z^* \vee y) \wedge z$, i.e. $x \ll y \wedge z$. Applying ($\ll$3), we obtain, finally, that $x \ll y$. So, $\ll = \ll_M$. Therefore $f \circ g = id_{\mathcal{MA}}$.

We have proved that f and g are bijections. □

Proposition 6.4 *Let L be a locally compact Hausdorff space. Then*

$$(RC(L), \subseteq, \ll_L),$$

where, for all $F, G \in RC(L)$*,* $F \ll_L G$ *iff* F *is compact and* $F \subseteq \mathrm{Int}(G)$*, is a MVD-algebra. All such MVD-algebras will be called* standard MVD-algebras*.*

Proof. It is straightforward to verify that the MVD axioms hold. Axiom ($\ll$5) is the most tricky. It follows from the established fact that: for every compact subspace A of a locally compact space L and every open set $V \subseteq L$ that contains A there exists an open set $U \subseteq L$ such that $A \subseteq U \subseteq \mathrm{cl}(U) \subseteq V$ and $\mathrm{cl}(U)$ is compact (see Theorem 3.3.2 in [16]). $\square$

Theorem 6.5 (Main Theorem)

(a) Each MVD-algebra $(W, \leq, \ll)$ *can be embedded into a standard MVD-algebra* $(RC(L), \subseteq, \ll_L)$*, where* L *is a locally compact Hausdorff space. When* W *is complete this embedding becomes a complete isomorphism.*

(b) There exists a bijective correspondence between the class of all (up to isomorphism) complete MVD-algebras and the class of all (up to homeomorphism) locally compact Hausdorff spaces.

Proof. We have, by Theorem 6.3, that the function $g : \mathcal{MA} \longrightarrow \mathcal{LCA}$, where $g((W, \leq, \ll)) = (W, \zeta, \mathbb{B})$, is a bijection. Moreover, in the proof of Theorem 6.3, we have shown that $x \ll y$ iff $x \in \mathbb{B}$ and $x \ll_\zeta y$, where $x \ll_\zeta y$ iff $x(-\zeta)y^*$. Now all follows from Theorem 5.14. $\square$

7 Concluding remarks

In this paper we have demonstrated the usefulness of the theory of proximity spaces to some region-based theories of space. Defining connection in terms of proximity gives a new semantics for this relation which has not previously been used. The standard meaning of the connection between regions is not always suitable to describe the spatial configuration between them. For instance, the closed regions in the topological space $\mathbb{Q}$ of rational numbers have the same spatial nature as those in the space $\mathbb{R}$ of real numbers, but in $\mathbb{Q}$ the regions $A = \{x : 1 \leq x^2 \leq 2\}$ and $B = \{x : 2 \leq x^2 \leq 4\}$, for example, are not standardly connected, because they do not share a common point, while in $\mathbb{R}$ they are standardly connected. This is because $\mathbb{Q}$ does not have enough points. If we consider $\mathbb{Q}$ as a proximity space, defined by the natural metric d in $\mathbb{Q}$ (see Example 4 in 3.4), then we have $A\delta B$, because $d(A, B) = 0$, so they are connected by the proximity definition of the connection relation. According to the Representation theorem for connection algebras, proximity connection algebras describe the same spatial picture as the corresponding standard ones. This shows that proximity semantics and the standard semantics for the connection relation are in a sense equivalent.

The representation theorems for connection algebras and local connection algebras, which we have presented in the paper, show that the corresponding axioms indeed characterize the intended spatial properties of regions. The system of connection algebra is the simplest one and deals only with the Boolean nature of the regions and the connection relation. The local connection algebras (Roeper's region-based topologies) take into account another important primitive notion of spatial regions: boundedness (or "limitedness", according to Roeper's terminology). It was shown that both formalizations can be studied successfully by the machinery of proximity spaces. The equivalence of MVD-algebras with local connection algebras shows that such notions as the connection relation and boundedness can be incorporated into a single mereological in nature relation, namely the modified Mormann's interior parthood. This enables a Whiteheadian theory of space to be axiomatized in terms of a single mereological relation, which was a main purpose of the Mormann's paper [28].

As a future work we hope to extend our representation theorems for the region connection calculus RCC. Note that the notion of Boolean Connection Algebra (BCA) in [34] is just another equivalent formulation of RCC. There are two difficulties here. The one is that BCA do not satisfy the axiom (C5) which is essential in the applications of proximity spaces. Note that the system of Grzegorczyk does not contain this axiom, but to obtain a system with topological representation theorem he uses non-elementary axioms, containing the definable notion of a point. The second difficulty is that BCA satisfy the following axiom, which is not in our systems:

(∗) If $x \neq 1$ and $x \neq 0$ then xCx^*.

This axiom is satisfied in connected topological spaces. The difficulty in our representation construction for connection algebras is that this axiom does not entail connectedness of the obtained topological space.

Acknowledgements. We are very grateful to the referee for the useful remarks and suggestions.

References

[1] Bennett, B. A categorical axiomatisation of region-based geometry. *Fundamenta Informaticae 46(1–2)* (2001), 145–158.

[2] Bennett, B., and Cohn, A. G. and Torrini, P. and Hazarika, S. M. A Foundation for Region-Based Qualitative Geometry. *Proceedings of ECAI-2000, ed. W. Horn* (2000), 204–208.

[3] Biacino, L., and Gerla, G. Connection structures. *Notre Dame Journal of Formal Logic 32* (1991), 242–247.

[4] Biacino, L., and Gerla, G. Connection structures: Grzegorczyk's and Whitehead's definition of point. *Notre Dame Journal of Formal Logic 37* (1996), 431–439.

[5] Čech, E. *Topological Spaces.* Interscience, London, 1966.

[6] Choquet, G. Sur les notions de filtre et de grille. *Comptes Rendus Acad. Sci. Paris 224* (1947), 171–173.

[7] Clarke, B. L. A calculus of individuals based on 'connection'. *Notre Dame Journal of Formal Logic 22* (1981), 204–218.

[8] Clarke, B. L. Individuals and points. *Notre Dame Journal of Formal Logic 26* (1985), 61–75.

[9] Cohn, A., and Hazarika, S. Qualitative spatial representation and reasoning: An overview. *Fundamenta Informaticae 46* (2001), 1 – 29.

[10] de Laguna, T. Point, line and surface as sets of solids. *The Journal of Philosophy 19* (1922), 449–461.

[11] Deneva, A., and Vakarelov, D. Modal logics for local and global similarity relations. *Fundamenta Informaticae 31* (1997), 295–304.

[12] Düntsch, I., Orlowska, E., and Wang, H. Algebras of approximating regions. *Fundamenta Informaticae 46* (2001), 71 – 82.

[13] Düntsch, I., Schmidt, G. and Winter, M. A Necessary Relation Algebra for Mereotopology. *Studia Logica*, 69(2001), 381–409.

[14] Düntsch, I., Wang, H., and McCloskey, S. A relation algebraic approach to the Region Connection Calculus. *Theoretical Computer Science 255* (2001), 63–83.

[15] Düntsch, I., Wang, H., and McCloskey, S. Relation algebras in qualitative spatial reasoning. *Fundamenta Mathematicae* 39(1999), 229–248.

[16] Engelking, R. *General Topology.* PWN, Warszawa, 1977.

[17] Efremovič, V. Infinitesimal spaces. *DAN SSSR 76* (1951), 341–343.

[18] Efremovič, V. The geometry of proximity I. *Mat Sbornik (New Series) 31* (1952), 189–200. In Russian.

[19] Gerla, G. Pointless geometries. In *Handbook of Incidence Geometry*, F. Buekenhout, Ed. Eslevier Science B.V., 1995, ch. 18, pp. 1015–1031.

[20] Grzegorczyk, A. Axiomatization of geometry without points. *Synthese 12* (1960), 228–235.

[21] Hayes, P. J. The second naive physics manifesto. In *Formal Theories of the Commonsense World.* Ablex Publishing Corp., Norwood, New Jersey, 1984.

[22] Hu, S.-T. Boundedness in a topological space. *J. Math. Pures Appl. 28* (1949), 287–320.

[23] Koppelberg, S. Some classes of projective Boolean algebras. *Math. Annalen 201* (1973), 283–300.

[24] Koppelberg, S. *General Theory of Boolean Algebras*, vol. 1 of *Handbook on Boolean Algebras.* North Holland, 1989.

[25] Leader, S. Local proximity spaces. *Math. Annalen 169* (1967), 275–281.

[26] Leonard, H. S, and Goodman, N. The calculus of individuals and its uses. *Journal of Symbolic Logic, 5* (1940), 45–55.

[27] Leśniewski, S. Podstawy ogólnej teoryi mnogosci.I. *Prace Polskiego Kola Naukowe w Moskwie, Sekcya matematycznoprzyrodnicza 2* (1916).

[28] Mormann, T. Continuous lattices and Whiteheadian theory of space. *Logic and Logical Philosophy 6* (1998), 35–54.

[29] Naimpally, S. A., and Warrack, B. D. *Proximity Spaces.* Cambridge University Press, Cambridge, 1970.

[30] Randell, D. A., Cohn, A. G., and Cui, Z. Computing transitivity tables: A challenge for automated theorem provers. In *Proceedings of the 11th International Conference on Automated Deduction (CADE-11)* (Saratoga Springs, NY, June 1992), D. Kapur, Ed., vol. 607 of *LNAI*, Springer, pp. 786–790.

[31] Roeper, P. Region-based topology. *Journal of Philosophical Logic 26* (1997), 251–309.

[32] Sikorski, R. *Boolean Algebras.* Springer-Verlag, Berlin, 1964.

[33] Smirnov J. M. On proximity spaces. *Mat. Sb. 31* (1952), 543–574. In Russian.

[34] Stell, J. Boolean connection algebras: A new approach to the Region Connection Calculus. *Artificial Intelligence 122* (2000), 111–136.

[35] Stell, J., and Worboys, M. F. The algebraic structure of sets of regions. In *Proceedings of the 3rd International Conference on Spatial Information Theory (COSIT 97)* (1997), S. C. Hirtle and A. Frank, Eds., vol. 1329 of *Lecture Notes in Computer Science*, Springer–Verlag, pp. 163–174.

[36] Šwarc, A. S. Proximity spaces and lattices. *Ucen. Zap. Ivanovsk Gos. Ped. Inst. 10* (1956), 55–60. In Russian.

[37] Thron, W. J. Proximity structures and grills. *Math. Ann. 206* (1973), 35–62.

[38] Tarski, A. Les fondements de la gèomètrie des corps. First Polish Mathematical Congress, Lwø'w, 1927. English translation in J. H. Woodger (Ed.) (1956) in: *Logic, Semantics, Metamathematics*, Clarendon Press.

[39] Vakarelov, D. Information systems, similarity relations and modal logics. In *Incomplete Information – Rough Set Analysis*, E. Orlowska, Ed. Physica – Verlag, Heidelberg, 1997, 492–550.

[40] Vakarelov, D. Proximity modal logics. In *Proceedings of the 11th Amsterdam Colloquium* (December 1997), 301–308.

[41] Vakarelov, D., Düntsch, I. & Bennett, B. A note on proximity spaces and connection based mereology. In C. Welty & B. Smith (Eds.), *Proceedings of the 2nd International Conference on Formal Ontology in Information Systems (FOIS'01)*, 2001, 139–150.

[42] Varzi, A. C. Parts, wholes, and part–whole relations: The prospect of mereotopology. *Data & Knowledge Engineering 20* (1996), 259–286.

[43] Whitehead, A. N. *Process and reality.* MacMillan, New York, 1929.

Dimiter Vakarelov
Department of Mathematical Logic
with Laboratory for Applied Logic,
Faculty of Mathematics and
Computer Science, Sofia University,
Blvd. James Bouchier 5, 1126 Sofia, Bulgaria
dvak@fmi.uni-sofia.bg

Georgi Dimov
Faculty of Mathematics and
Computer Science,
Sofia University,
Blvd. James Bouchier 5,
1126 Sofia, Bulgaria
gdimov@fmi.uni-sofia.bg

Ivo Düntsch
Department of Computer Science,
Brock University, St. Catherines,
Ontario, Canada, L2S 3AI
duentsch@tarski.org

Brandon Bennett
School of Computing
University of Leeds
Leeds LS2 9JT, England
brandon@comp.leeds.ac.uk