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Mereotopology in 2nd-Order and Modal Extensions of Intuitionistic Propositional Logic

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Abstract

We show how mereotopological notions can be expressed by extending intuitionistic propositional logic with propositional quantification and a strong modal operator. We first prove completeness for the logics wrt Kripke models; then we trace the correspondence between Kripke models and topological spaces that have been enhanced with an explicit notion of expressible region (r-spaces). We show how some qualitative spatial notions can be expressed in topological terms. We use the semantical and topological results in order to show how in some extensions of the logics it is possible to express connectedness, non-emptiness and a set of jointly exhaustive, pairwise disjoint, binary relations that play a significant role in qualitative spatial reasoning.

1 Introduction

Logic-based spatial representations and qualitative spatial languages are often investigated in relation to common-sense reasoning, geographical information systems and computer graphics [CBGG97, SW99, PL97].

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One of the main motivations is to introduce models of space that are based on an intuitively natural ontology of spatial regions in a fixed dimension. One feature often required of such an ontology is that all configurations of regions that can be consistently described in the language can be exemplified without recourse to boundary lines that display complexity on an infinitesimally small scale. The expressive richness of the standard model of space, regarded as a Cartesian product of the reals, makes this kind of intuition difficult to handle, since seemingly natural ways to define regions in the standard metric topology of $\mathcal{R}^n$ force the introduction of concretely infeasible elements [PS98].

An idea that goes back to Whitehead and Tarski, and has lately received considerable attention in AI, is that of taking qualitatively significant relations between region as primitives, and axiomatise them according to intuition ([Ger95] for an historical survey). These relations are quite often those of *part of* and *connection* [Leś31, Sim87]. Relevant examples of systems based on this approach, often labelled as *mereotopological*, have been presented in [Cla81] and more recently in [RCC92] (system RCC). The relationship that mereotopology has with algebraic, point-free topology [FS79, Vic89], naturally arising out of Kuratowski's axioms [MT44], has been highlighted in recent work [Ste00, Mor98]. Mereotopology also bears some relationship with logics that have topological semantics, such as intuitionistic logic and *S4* [RS63], as well as with modal logics where semantical relations can be considered spatially [BDCTV97, LP96]. Topological semantics have been used in [Ben96, She99] to give logical encodings of spatial notions.

Altogether, there has been considerable attention for the use of non-classical logic in qualitative spatial reasoning, and for the idea of a *spatial* interpretation of a logic. Among others, [LP96, PS98] address the issue of what ultimately matters for a logic to be "spatial" in a qualitative sense; since this essentially amounts to modelling a connection relation that matches intuition in a space of fixed dimension, having a topological semantics is not generally enough.

The approach that we are following here is that of building on top of the topological semantics of intuitionistic logic first introduced in [Tar56], where models are based on topological spaces, and formulæ are interpreted as open sets. This idea has already been used by researchers investigating RCC; a fragment of that system (often referred to as $RCC8$) has been encoded in intuitionistic logic, in order to get a better understanding of its computational properties [Ben96]; Heyting algebras have been used in order to gain a better model-theoretical understanding of the whole calculus [Ste00].

The open sets that are interpretation of some formula can be taken, from the point of view that we are presenting here, as the *regions* that can be expressed in our language; quite naturally, these might well not include all the open sets, which are often uncountable, since expressions in a language are normally taken to be countable. A bit more precisely, yet still informally, we can associate our "atomic formulæ" (a subset of the propositional variables) to regions that are, semantically, the primitive ones; then, the intuitionistic operators can be interpreted as the topological constructors (including union, intersection and Heyting complement); this will keep us at the propositional level.

We are going to investigate the gain in expressive power that can be obtained by extending propositional intuitionistic logic (used in [Ben96]) with propositional quantification (ie, shifting to intuitionistic 2nd-order propositional logic, ISPL) and with a "strong" modal operator. Both ISPL and intuitionistic modal logic are, separately taken, well-established research topics [Pra65, Bar92, WZ95]. The use of modality that we are presenting here is not really the standard one, thought, since it is intended to capture some non-constructive aspects that are needed in order to express things like "the region A is not empty in the model".

We are going to consider two forms of 2nd-order propositional logic, *C2h* (already introduced in [Gab74]) and *2At* (a new one) together with their modal extensions N*C2h* and N*2At*. We are essentially going to examine the expressiveness of the language in the different logics, with respect to a set of mereotopological relations, inclusive of the so-called $RCC8$ relations [RCC92], the part-whole relation (corresponding to point-wise inclusion between sets), connectedness (the property of being made of one piece), connection (the relation between two regions that are either overlapping or adjacent) and non-emptiness. Some results are stated in lemmas 16, 26 and in theorems 5, 6, 7 and 9. These results are technically quite straightforward, given the completeness of the logics wrt Kripke models (theorems 1, 2, 3, 4), the correspondence with topological models (lemmas 16 and 17), and the way in which the restrictions affect expressiveness (lemmas 19, 22).

An interesting technical issue is the way in which 2nd-order propositional quantification can be used to represent quantification over regions. Given a topological space $(S, \mathcal{O})$ (where $\mathcal{O}$ are the open subsets on a set of points S) the open sets that are expressible in a model will be intended to be the regions in that model, forming a collection $\mathcal{R} \subseteq \mathcal{O}$, inclusive at least of the empty set, of the whole space, and of all those sets that can be defined using the constructors (corresponding to the logical operators, as in def.12 and lemma 16). The case in which every open set is expressible ($\mathcal{R} = \mathcal{O}$) is just a limit case, giving what we can call a *principal* model (after [Kre97]). A quite natural order of expressiveness could then be associated to the class of the models that are defined on the same space.

According to the semantics that we are going to present, the fact that some relation R holds between some regions $A_1, \ldots, A_n$ in a model, can be expressed by the validity, in that model, of a formula $\gamma_R(\alpha_1/x_1, \ldots, \alpha_n/x_n)$ (syntactical representation of the relation), in which some formulae $\alpha_1, \ldots, \alpha_n$ that are respectively interpreted as $A_1, \ldots, A_n$ are uniformly replaced for $x_1, \ldots, x_n$ (the free propositional variables occurring in γ_R).

There are different ways in which it is possible to represent mereotopological notions, in our 2nd-order intuitionistic modal language. One possibility is that of introducing extra axioms and, correspondingly, restrictions on the models; this allows us getting simple definitions for some of the new notions. For example, a form of connectedness is expressible in an axiomatic extension of intuitionistic 2nd-order propositional logic, without any use of modality (thm. 5), introducing a restriction that can be expressed in the language (def.20, lemma 28). Adding modal schemas that enforce further restrictions (definitions 23, 19, lemmas 30, 31) all the RCC8 relations become expressible (theorems 7, 9); in particular, this is possible using expressions that do not contain the modal operator (thm. 9). The restrictions on the topological spaces that match the specific axioms are quite strong, but still lend to an interpretation that is relevant from the point of view of the representation of geographical information.

Another possibility is that of introducing the new notions using only definitions, relying on syntactical representations that may contain both the propositional quantifier and the modal operator; here we give an example related to connectedness (lemma 26), otherwise leaving this approach for further development.

Ultimately, a natural idea of region, as something that can be described in terms of a fixed dimensionality, without reference to lower dimensional features (like 1-dimensional cracks in a 2-dimensional surface), matches just those regions that are topologically *regular*; that is, expressible open sets that are equivalent to the interior of their closure. Regularisation can be represented syntactically, using double negation. Some of the regions that are not regular can be used, however, to represent relations between the regular ones.

Besides regularity, there are other intuitive demands on the "natural" regions, depending essentially on the behaviour of the connection relation [LP96], and essentially directed to ban regions that are either incompatible with some fixed dimension constraint, or too convoluted to be drawn schematically. The constraints depending on dimensionality could be dealt with referring to an approach based on Kuratowski's theorem (this has been used in [PS98]), but shall not be pursued any further in this paper. On the other hand, constraints ruling out regions that, even in a finite domain of regions of fixed, finite dimension, could not be represented schematically using a finite number of linear traits to draw their boundaries, can be associated to what we have called *well-connectedness*, and addressed in subsection 5.3.

Topological spaces, similar to the ones that we are using in one significant respect (being order topologies), and a representation of connection relations that bear some relation to those we are discussing, can be found in [Kop92, Kov92], where a general approach to image processing based on so-called *digital spaces* is presented.

Differently from RCC [RCC92], expressing connection in our framework is compatible with the existence of regions that are minimal, ie not containing any smaller, non-empty subregions. In models based on *2At* (but not in those based on *C2h*) every region must indeed contain a minimal one (def. 4); this, as a topological property of the model, shall be called here *terminability* (subsection 4.1). Such a property is relevant, if not sufficient, in view of a possible restriction to finite models, that could make a logic interesting wrt application in the verification of topological constraints.

2 ISPL

An intuitionistic 2nd-order propositional logic (ISPL) is obtained by adding to intuitionistic propositional logic (IPL) some form of quantification on the propositional variables; this can actually be done in different senses and ways. Of the several existing definitions, some rely on the semantics [Kre97, Skv97], some other ones rely either on

Hilbert axiomatisations [Gab74, Gab81] or on rule systems [Loe76].

According to an interpretation which is primarily semantical, quantification ranges on all the possible *denotations* of the propositional variables; these can be regarded as the semantical propositions, whatever the style of the semantics.

On the other hand, according to an essentially syntactical interpretation, quantification ranges over all possible *substitutions* of propositional variables. Unless we ensure that the language includes a variable referencing every possible semantic denotation, the range of substitutional quantification may be a proper subset of the semantical domain.

The gap between the two definitions is a deep one: [Kre97, Skv97] show that the logic with semantical quantification is not recursively axiomatisable, whereas logics with syntactical quantification can be axiomatised in a form very similar to that of intuitionistic 1st-order predicate calculus (*IPC*).

2.1 Syntax for ISPL

Axiomatisable ISPLs can be formulated in a propositional language $\mathcal{L}$, where $Var = x, y, \ldots$ are propositional variables and $\rightarrow, \forall$ are the primitive operators. The set of the formulæ *Wff* is as usual the smallest one including *Var* and closed wrt the operators. $\alpha, \beta, \ldots$ are metavariables for formulæ, and $\alpha(x)$ means that x may occur in α. The expression $\alpha(\beta/x)$ denotes that the formula β is uniformly substituted for the variable x, ie, every occurrence of x in α (possibly none) is replaced by an occurrence of β, avoiding any capture of free variables by renaming. The expression $\alpha[\beta/\gamma]$ will be used, somewhat informally, to denote that some occurrences (possibly none) of γ in α are replaced by occurrences of β. The propositions (*Prop*) are formulæ without free variables. The connective $\rightarrow$ is left-associative; precedence is $\{\sim, \approx, \bigwedge_{1<i<n}, \bigvee_{1<i<n}\} > \{\wedge, \vee\} > \rightarrow > \leftrightarrow > \{\exists, \forall\}$, where the remaining operators are defined as follows.

Def. 1 $\alpha \wedge \beta = \forall x.(\alpha \rightarrow \beta \rightarrow x) \rightarrow x$

$\alpha \vee \beta = \forall x.(\alpha \rightarrow x) \rightarrow (\beta \rightarrow x) \rightarrow x$

$\exists x.\alpha(x) = \forall z.(\forall x.\alpha(x) \rightarrow z) \rightarrow z$ (z not free in α)

$\bot = \forall x.x$

$\sim \alpha = \alpha \rightarrow \bot$

$\top = \bot \rightarrow \bot$

$\alpha \leftrightarrow \beta = (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$

$\approx\alpha = \sim\sim\alpha$
$\bigwedge_{0<i<n}\alpha_i = \alpha_1 \wedge \ldots \wedge \alpha_{n-1}$
$\bigvee_{0<i<n}\alpha_i = \alpha_1 \vee \ldots \vee \alpha_{n-1}$

Hilbert axiomatisations for different versions of ISPL, among which *C2h* and *2At*, can be obtained from the following, assuming uniform substitution for free variables.

A1. $\vdash \alpha \to \beta \to \alpha$
A2. $\vdash (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma$
A3. $\vdash (\forall x.\alpha(x)) \to \alpha(y/x)$
A4. $\vdash (\forall x.\alpha \vee \beta(x)) \to \alpha \vee (\forall x.\beta(x))$ $\quad$ x not free in α.
A5. $\vdash (\forall x.\approx\alpha(x)) \to \approx\forall x.\alpha(x)$
A6. $\vdash \exists x.x \leftrightarrow \alpha$ $\quad$ α any formula, x not free in α.
A7. $\vdash \alpha$ and $\vdash \alpha \to \beta$ implies $\vdash \beta$
A8. $\vdash \alpha \to \beta(y/x)$ implies $\vdash \alpha \to \forall x.\beta(x)$ $\quad$ x not free in α

Axiom schemas A1, A2 and rule A7 by themselves axiomatise positive implication; the schema A3 and rule A8 axiomatise standard quantification; schema A6 is a full comprehension principle for 2nd-order propositional logic; these together give an axiomatisation for a weak form of 2nd-order intuitionistic logic [Gab74].

Schemas A1–A4 with A6 and rules A7–A8 give the axiomatisation of *C2h*, the intuitionistic 2nd-order propositional logic with constant domain presented in [Gab81] (*C2I* in [Gab74]).

With or without A4, full comprehension can be dropped and weaker logics obtained; in those case, though, independent axioms for $\wedge, \vee, \exists, \bot$ are needed.

Here we introduce *2At* as a logic stronger than *2Ch*, that can be axiomatised with schemas A1–A6 and rules A7–A8. The schema A5 has been already considered for an extension of intuitionistic predicate calculus in [Gab81] (the logic that there is called *MH*).

2.2 Modal extensions

Let us extend $\mathcal{L}$ to the language $\mathcal{L}_m$ by adding the construct $\Box\,\alpha$, where $\Box$ is a modal operator of necessity, taken as an extra primitive. We now consider the modal extensions of *C2h* and *2At*, respectively *NC2h* and N*2At*, obtained by adding the following definition and postulates (possibly, with some redundancy).

Def. 2 $\Diamond\alpha = \sim\Box\sim\alpha$;

A9. $\vdash \Box\alpha \to \alpha$
A10. $\vdash \Box\alpha \to \Box\Box\alpha$
A11. $\vdash \Box(\alpha \to \beta) \to \Box\alpha \to \Box\beta$
A12. $\vdash (\forall x.\Box\alpha(x)) \to \Box\forall x.\alpha(x)$
A13. $\vdash \Box(\alpha \vee \beta) \to \Box\alpha \vee \Box\beta$
A14. $\vdash (\Box\exists x.\alpha(x)) \to \exists x.\Box\alpha(x)$
A15. $\vdash \Box\alpha \vee \sim\Box\alpha$
A16. $\vdash \alpha$ implies $\vdash \Box\alpha$

Schemas 9–11 and rule 16 are those generally used to axiomatise an $S4$-style necessity operator [Pra65]. Schema 12 (Barcan formula) is also quite standard in quantified modal logics. Schemas 13–15 are more specific, and they are needed in order to interpret $\Box$ as a "strong" modality in our models (in fact, very similar to ordinary Kripke models for intuitionistic logic). Due to the presence of rule 16 and schema 15, formulæ where each propositional variable is in the scope of an occurrence of $\Box$, have a behaviour that can mimic the behaviour of formulæ in classical logic. On the other hand, the definition of $\Diamond$ in terms of intuitionistic negation gives that the duality with $\Box$ is lost.

2.3 Metatheorems

For each logic L of *2Ch*, N*2Ch*, *2At*, N*2At*, a deduction of α from a set of premises Δ (also $\Delta \vdash_L \alpha$) is defined inductively as a finite sequence of formulæ (steps), such that each step is justified either as an axiom, as an assumption, or as the conclusion of a rule application to previous steps, with the following proviso(s):

1. A8 cannot be applied in any way that binds x, if x is free in the premises.

2. (Modal logics only). A16 can be applied only if, whenever a formula α is in the premises, and it has not form $\Box\beta$, then also $\Box\alpha$ is in the premises.

Lemma 1 (Deduction theorem) $\Delta, \alpha \vdash_L \beta$ iff $\Delta \vdash_L \alpha \to \beta$.

Proof. *Right to left.* By $\Delta, \alpha \vdash_L \alpha$ and rule A7.

Left to right. By induction on the length of the deduction.

Base case. If β is an axiom, by A1 and rule A7. If $\beta \in \Delta$, by $\vdash_L \beta \to \beta$.

Step case. (a) For an application of A7, by induction hyp., A2 and A7 itself.

(b) For an application of rule A8, we can start from the induction hyp., then convert $\Delta \vdash_L \alpha \to \beta \to \gamma$ into the equivalent $\Delta \vdash_L \alpha \wedge \beta \to \gamma$, and apply rule A8, remembering proviso 1.

(c) - only for modal logics. For an application of rule A16, we can start from the induction hyp., then apply rule A16 (proviso 2 must be satisfied, by the hyp. on the original deduction), obtaining a deduction of form $\Delta \vdash_L \Box(\alpha \to \beta)$; using A11 with rule A7 we get $\Delta \vdash_L \Box\alpha \to \Box\beta$. Now, if α has form $\Box\gamma$; then $\Delta \vdash_L \alpha \to \Box\beta$ follows, using A10. Else, by proviso 2, $\Box\alpha \in \Delta$; then, using rule A7, we get $\Delta \vdash_L \Box\beta$, and so $\Delta \vdash_L \alpha \to \Box\beta$ using A1 and rule A7.

Lemma 2 (Replacement of equivalents)

$\Delta \vdash_L \alpha \leftrightarrow \beta$ implies $\Delta \vdash_L \gamma[\alpha/x] \leftrightarrow \gamma[\beta/x]$

where the occurrences of x that are replaced are intended to be the same on both sides of $\leftrightarrow$.

Proviso (*) (modal logics only): if γ contains some occurrence of $\Box$, then for any $\phi \in \Delta$ s.t. ϕ has not form $\Box\eta$, also $\Box\phi \in \Delta$.

Proof. By induction on the length of the formulæ. *Base case.* If $\gamma = x$, by assumption. If x is not free in γ, by $\vdash_L \gamma \to \gamma$.

Step case. (a) $\gamma = \eta \to \delta$. As consequences of the induction hypothesis, $\Delta \vdash_L \eta[\alpha/x] \leftrightarrow \eta[\beta/x]$ and $\Delta \vdash_L \delta[\alpha/x] \leftrightarrow \delta[\beta/x]$; since

$\eta[\beta/x] \to \eta[\alpha/x], \delta[\alpha/x] \to \delta[\beta/x] \vdash_L (\eta[\alpha/x] \to \delta[\alpha/x]) \to (\eta[\beta/x] \to \delta[\beta/x])$

it follows $\Delta \vdash_L (\eta[\alpha/x] \to \delta[\alpha/x]) \to (\eta[\beta/x] \to \delta[\beta/x])$.

Similarly for $\Delta \vdash_L (\eta[\beta/x] \to \delta[\beta/x]) \to (\eta[\alpha/x] \to \delta[\alpha/x])$.

(b) $\gamma = \forall y.\delta(y)$. Then, for any z not free in Δ, as consequences of the ind. hyp., $\Delta \vdash_L \delta(z/y)[\alpha/x] \to \delta(z/y)[\beta/x]$;

using A3 it follows $\Delta \vdash_L (\forall z.\delta(z/y)[\alpha/x]) \to \delta(z/y)[\beta/x]$; then, by rule A8, $\Delta \vdash_L (\forall z.\delta(z/y)[\alpha/x]) \to (\forall z.\delta(z/y)[\beta/x])$. Similarly for the other side of the arrow.

(c) - modal logics only: $\gamma = \Box\delta$. As consequences of the ind. hyp., $\Delta \vdash_L \delta[\alpha/x] \to \delta[\beta/x]$; since by hyp. proviso (*) holds, rule A16 can be applied satisfying the modal proviso in lemma 1, giving a deduction $\Delta \vdash_L \Box(\delta[\alpha/x] \to \delta[\beta/x])$; then, using A11, $\Delta \vdash_L \Box\delta[\alpha/x] \to \Box\delta[\beta/x]$. Similarly for $\Delta \vdash_L \delta[\beta/x] \to \delta[\alpha/x]$.

The following gives an example of a non-theorem in *C2h* which is provable in *2At*:

Lemma 3 $\vdash_{2At} \approx\forall x.\alpha(x) \vee \sim\alpha(x)$

Proof. Follows from $\vdash_{C2h} \forall x.\approx(\alpha(x) \vee \sim\alpha(x))$ using A5 and rule A7.

3 Kripke semantics

In [Gab74, Gab81] Kripke semantics for *C2h* and for *MH* are given. In order to give a semantics for *2At*, Gabbay's models for *C2h* must be modified, adding a specific restriction closely related to that required by *MH*. Besides, we need to add some machinery for modality.

Def. 3 A *2At*-frame is a structure $\mathcal{F} = (S, \leq, \mathbf{0})$, where $(S, \leq)$ is a partial order, with a minimum $\mathbf{0} \in S$ (the *root*), on the set of points S. $\mathcal{U}_{\leq}$ is the class of the subsets of S that are upper-closed wrt $\leq$.
Furthermore, $(S, \leq)$ satisfies the following condition (condition *mh*): for any $\mathtt{x} \in S$, there is $\mathtt{y} \in S$, such that $\mathtt{x} \leq \mathtt{y}$, and for all $\mathtt{z} \in S$, if $\mathtt{y} \leq \mathtt{z}$ then $\mathtt{z} \leq \mathtt{y}$; we then say that $\mathtt{y}$ is *terminal*.
Dropping condition *mh* we have a *C2h*-frame (as in [Gab81]).

Condition *mh* on the frames corresponds to schema A5 [Gab81].

Def. 4 A Kripke N*2At*-model (N*C2h*-model) is a triple $\mathcal{M} = (\mathcal{F}, \mathcal{R}, \rho)$, where $\mathcal{S} = (S, \leq, \mathbf{0})$ is a *2At*-frame (*C2h*-frame), ρ is an interpretation, assigning to each $x \in Var$ an element $\|x\|_\rho \in \mathcal{R} \subseteq \mathcal{U}_{\leq}$, and $\mathcal{R}$ is the image of ρ.
We inductively extend the interpretation to all the formulæ, defining the truth of α at a point $\mathtt{a}$, $\mathtt{a} \vDash \alpha$, and adding a comprehension condition, as follows.

1. For $\alpha \in Var$, $\mathtt{a} \vDash \alpha$ iff $\mathtt{a} \in \|\alpha\|_\rho$.
2. $\mathtt{a} \vDash \alpha \to \beta$ iff for every $\mathtt{b}$ such that $\mathtt{a} \leq \mathtt{b}$, whenever $\mathtt{b} \vDash \alpha$ then $\mathtt{b} \vDash \beta$.
3. $\mathtt{a} \vDash \forall x.\alpha(x)$ iff for every $y \in Var$, $\mathtt{a} \vDash \alpha(y/x)$.
4. $\mathtt{a} \vDash \Box\alpha$ iff $\mathbf{0} \vDash \alpha$.
5. Full Comprehension (condition *fc*): for each $\alpha \in Wff$, there is $x \in Var$ s.t. $\mathtt{a} \vDash \alpha$ iff $\mathtt{a} \in \|x\|_\rho$.

For any formula α, we refer to $\|\alpha\|_\rho = \{\mathtt{a} \in S : \mathtt{a} \vDash \alpha\}$ as to its *truth-set*, in a model $\mathcal{M}$ as above.
Omitting $\Box$, we get a *2At*-model (*C2h*-model, as in [Gab81]).

Given full comprehension, it can be trivially shown that every truth-set is an upper-closed set that is a member of $\mathcal{R}$. We have also the following.

Lemma 4 (*Hereditary condition*) If $\mathtt{a} \leq \mathtt{b}$ then $\mathtt{a} \models \alpha$ implies $\mathtt{b} \models \alpha$.

Proof. Since every truth-set is an upper-closed set wrt $\leq$.

We say that a formula α is *valid in a model* $\mathcal{M}$, writing $\models_M \alpha$, iff $\mathbf{0} \models \alpha$ in that model; clearly, in this case the truth-set of α equals the whole of S.
We say that a formula α is *semantically deducible* from a set of formulæ $\Delta = \beta_{i \in I}$ in $\mathcal{M}$, and we write $\Delta \models_M \alpha$, iff for every point $\mathtt{a}$, whenever in $\mathcal{M}$ $\mathtt{a} \models \beta_i$ for every $i \in I$, then $\mathtt{a} \models \alpha$.

Lemma 5 The following can be proved, for any model, using the definitions of the logical operators and the interpretation rules:

1. For every $\mathtt{a} \in S$, $\mathtt{a} \not\models \bot$.
2. For every $\mathtt{a} \in S$, $\mathtt{a} \models \top$.
3. $\mathtt{a} \models \alpha \wedge \beta$ iff $\mathtt{a} \models \alpha$ and $\mathtt{a} \models \beta$.
4. $\mathtt{a} \models \alpha \vee \beta$ iff $\mathtt{a} \models \alpha$ or $\mathtt{a} \models \beta$.
5. $\mathtt{a} \models \exists x.\alpha(x)$ iff $\mathtt{a} \models \alpha(y/x)$ for some $y \in \mathit{Var}$.
6. $\mathtt{a} \models \sim\alpha$ iff for every $\mathtt{b}$, $\mathtt{a} \leq \mathtt{b}$, $\mathtt{b} \not\models \alpha$.
7. $\mathtt{a} \models \alpha \leftrightarrow \beta$ iff ($\mathtt{a} \models \alpha$ iff $\mathtt{a} \models \beta$).
8. $\mathtt{a} \models \sim\Box\alpha$ iff there exists $\mathtt{a} \in S$ s.t. $\mathtt{a} \models \alpha$.

Intuitively, $\mathcal{R}$ is the set of the upper-closed sets that are expressible in the model; given full comprehension, it always contain at least $\varnothing$ and S, and is closed with respect to the interpretation of each logical operator. We say that a model is *principal* iff $\mathcal{R} = \mathcal{U}_{\leq}$.

Full comprehension, corresponding to schema A6 [Gab74], introduces a character of impredicativity in the logic, ie, from the point of view of the inductive definition of formulæ, we get a dependency on the collection that is being defined; on the other hand, it gives quantification an intuitive character of completeness over expressible sets, and this is quite useful in order to define operators at the object level.

The interpretation rule for $\forall$ differs from the standard intuitionistic one not the least because we assume to have a quantification domain which is constant throughout the worlds, ie the set of the variables Var

does not change when we consider submodels. This assumption, corresponding to schema A4 [Gab81, Goe71], seems indeed quite natural in the propositional case.

The modal operator $\Box$ is interpreted as a *strong S5*-style necessity operator, so that the meaning of $\Box\alpha$ is that α is valid in the model; schemas A13–A14, under this interpretation, correspond to the fact that each model, wrt to validity, has the disjunction and existence property: ie, respectively, whenever $\models_M \alpha \vee \beta$, either $\models_M \alpha$ or $\models_M \beta$; and whenever $\models_M \exists x.\alpha(x)$, for some variable y of the language, $\models_M \alpha(y/x)$. On the other hand, from the presence of schema A15 immediately follows that provability cannot have the corresponding properties.

3.1 Completeness

We need to introduce a notion similar to that of *theory* in [Gab74, Gab81], relative to a logic L (*C2h*, *2At*, N*C2h*, N*2At*). We will often use $[\Delta, \alpha]$ as short for $\Delta \cup \{\alpha\}$.

Def. 5 A *k-theory* in a language $\mathcal{L}$ ($\mathcal{L}_m$), is a pair $(\Delta; \Omega)$ of sets of formulæ of $\mathcal{L}$ ($\mathcal{L}_m$).
$(\Delta; \Omega)$ is said to be *consistent* wrt a logic L iff for no finite subsets $\Delta' \subseteq \Delta, \Omega' \subseteq \Omega$, we have $\vdash_L \bigwedge\Delta' \rightarrow \bigvee\Omega'$.
$(\Delta; \Omega)$ is said to be *complete* in a language iff for all the formulæ α in a language, either $\alpha \in \Delta$ or $\alpha \in \Omega$.
$(\Delta; \Omega)$ is said to be *saturated* in a language wrt a logic L iff (a) $\Delta \vdash_L \alpha$ implies $\alpha \in \Delta$; (b) $\alpha \vee \beta \in \Delta$ implies $\alpha \in \Delta$ or $\beta \in \Delta$; (c) $\exists x.\alpha(x) \in \Delta$ implies that for some variable y in the language, $\alpha(y/x) \in \Delta$.
$(\Delta; \Omega)$ is said to be *of constant domains* in a language wrt a logic L iff whenever $(\Delta; [\Omega, \forall x.\alpha(x)])$ is consistent, then for some propositional variable y of the language, $(\Delta; [\Omega, \alpha(y/x)])$ is consistent.
$(\Delta; \Omega)$ is said to be a *ck-theory* in $\mathcal{L}$ ($\mathcal{L}_m$) iff it is a consistent, complete, saturated k-theory of constant domain in $\mathcal{L}$ ($\mathcal{L}_m$).
$(\Delta'; \Omega')$ is said to *extend* $(\Delta; \Omega)$ iff $\Delta \subseteq \Delta', \Omega \subseteq \Omega'$.

Consistent, complete, saturated k-theories of constant domains in a non-modal language $\mathcal{L}$ are used in the completeness proof for *C2h*.

Thm. 1 (Soundness and completeness for *C2h*).
For any formula α, $\vdash \alpha$ in *C2h* iff, for every *C2h*-model $\mathcal{M}$, $\models_M \alpha$ [Gab81].

The following notions are introduced in order to deal with terminability (condition *mh*) and modality.

Def. 6 We say that $(\Delta;\Omega)$ is *terminal* in $\mathcal{L}$ ($\mathcal{L}_m$) iff for every formula α in $\mathcal{L}$ ($\mathcal{L}_m$), either $\alpha \in \Delta$ or $\sim\alpha \in \Delta$.
A *r-theory* is a k-theory $(\Delta;\Omega)$ in a language $\mathcal{L}_m$ such that $\Box\alpha \in \Delta$ iff $\alpha \in \Delta$, and no other formulæ containing modality are in Δ.
$(\Delta;\Omega)$ is said to be a *root-theory* in $\mathcal{L}_m$ iff it is a consistent, complete, saturated k-theory of constant domain in $\mathcal{L}_m$, wrt a modal logic L, such that $\alpha \in \Delta$ iff $\Box\alpha \in \Delta$.
Given in $\mathcal{L}_m$ a root-theory $0 = (\Delta;\Omega)$, we say that a ck-theory $(\Delta';\Omega')$ is a 0-theory iff $\Delta \subseteq \Delta'$.

The modal logics we are considering have schema A15, so they cannot have themselves the disjunction property; however we are interested in models that have the corresponding semantical property; in order to restrict to such models, it is enough to consider, as the frames, only those partial orders that have a minimum; of course, no one of them can falsify all the non-theorems of a logic containing A15; this brings us close enough to a classical, non-constructive situation, though only for the modal formulae (those containing occurrences of $\Box$).

In the canonical models, points will be in general ck-theories. In the modal cases (N*C2h* and N*2At*) the root can be defined as a root-theory 0, whereas all the other points will be given as 0-theories. In the cases of *2At* and N*2At*, in order to satisfy condition *mh*, we need to include in the model, for any ck-theory (0-theory) a, a terminal ck-theory (0-theory) that extends a, playing the role as terminal element. We need first to prove some lemmas.

Lemma 6 Let $(\Delta;\Omega)$ be a consistent k-theory wrt L; then, for any formula α in the language, either $([\Delta,\alpha];\Omega)$ or $(\Delta;[\Omega,\alpha])$ is consistent.

Proof. Assume both $([\Delta,\alpha];\Omega)$ and $(\Delta;[\Omega,\alpha])$ are inconsistent. Then there must exist formulæ δ,ω such that (1) $\vdash_L \delta \wedge \alpha \rightarrow \omega$, (2) $\vdash_L \delta \rightarrow \omega \vee \alpha$, whereas (3) $\nvdash_L \delta \rightarrow \omega$. From (1) follows

$\vdash_L \alpha \to \delta \to \omega$, from this and (2) follows $\vdash_L \delta \to \omega \vee (\delta \to \omega)$, so $\vdash_L \delta \to (\delta \to \omega) \vee (\delta \to \omega)$, so $\vdash_L \delta \to \delta \to \omega$, and then $\vdash_L \delta \to \omega$, against (3).

Lemma 7 Let $(\Delta;\Omega)$ be a consistent r-theory in $\mathcal{L}_m$ wrt a modal logic L; then it can be extended to a root-theory $(\Delta';\Omega')$, in a language $\mathcal{L}'_m$ with possibly $\aleph_0$ more propositional variables.

Proof. We assume that $\alpha_1, \alpha_2, \ldots$ is an enumeration of the formulæ in $\mathcal{L}'_m$.

We define inductively a sequence of 0-theories $(\Delta_n;\Omega_n)$ such that for each n, $\Delta_n \subseteq \Delta_{n+1}$ and $\Omega_n \subseteq \Omega_{n+1}$.

Base case: $(\Delta_0;\Omega_0) = (\Delta;\Omega)$.

Step case: suppose $(\Delta_n;\Omega_n)$ is defined and consistent. We define $(\Delta_{n+1};\Omega_{n+1})$. There are two main cases.

A) $(\Delta_n;[\Omega_n,\alpha_n])$ is consistent. Then also $([\Delta_n, \sim\Box\alpha_n];[\Omega_n,\alpha_n])$ is consistent. In fact, assuming that it is not, since $([\Delta_n,\Box\alpha_n];[\Omega_n,\alpha_n])$ is inconsistent, using schema A15, we have that $(\Delta_n;[\Omega_n,\alpha_n])$ is inconsistent, against the hypothesis.

A1) $\alpha_n = \beta \to \gamma | \Box\beta$; then let $\Delta_{n+1} = [\Delta_n, \sim\Box\alpha_n]$ and $\Omega_{n+1} = [\Omega_n, \alpha_n]$.

A2) $\alpha_n = \forall x.\beta(x)$; then let $\Delta_{n+1} = [\Delta_n, \sim\Box\alpha_n, \sim\Box\beta(y/x)]$ and $\Omega_{n+1} = [\Omega_n, \alpha_n, \beta(y/x)]$, where y is the first new variable not used before. Also this gives a consistent theory (see [Gab74], lemma 1).

B) $\mathtt{p}_n = (\Delta_n;[\Omega_n,\alpha_n])$ is inconsistent.

Then let $\Delta_{n+1} = [\Delta_n, \alpha_n, \Box\alpha_n]$ and $\Omega_{n+1} = \Omega_n$.

We can prove that $(\Delta_{n+1};\Omega_{n+1})$ is consistent.

By construction, $\alpha \in \Delta_n$ iff $\Box\alpha \in \Delta_n$ (in the base case, this holds by definition of r-theory). Besides, since $\mathtt{p}_n$ is inconsistent, for appropriate δ, ω, such that δ is a conjunction of formulae in Δ and ω is a disjunction of formulae in Ω, we have $\vdash_L \delta \to \omega \vee \alpha_n$; then $\vdash_L \Box(\delta \to \omega \vee \alpha_n)$, by rule A16; then $\vdash_L \Box\delta \to \Box(\omega_1 \vee \alpha_n)$, using schema A11; then $\vdash_L \Box\delta \to \Box\omega_1 \vee \Box\alpha_n$, using A13; then $\vdash_L \Box\delta \to \omega_1 \vee \Box\alpha_n$. It follows that $(\Delta_n;[\Omega_n,\Box\alpha_n])$ is inconsistent; so, by lemma 6, $([\Delta_n,\Box\alpha_n];\Omega_n)$ is consistent.

Then, let $\Delta' = \bigcup_{n\in N} \Delta_n$ and $\Omega' = \bigcup_{n \in N} \Omega_n$. By construction, $\mathtt{q} = (\Delta';\Omega')$ is a consistent, complete, saturated k-theory of constant domain that extends $(\Delta;\Omega)$ in $\mathcal{L}'_m$ wrt L, and $\alpha \in \Delta_n$ iff $\Box\alpha \in \Delta_n$; so, $\mathtt{q}$ is a root-theory.

Lemma 8 Given a language $\mathcal{L}_m$, a modal logic L and a root-theory $0 = (\Delta;\Omega)$, for any consistent k-theory $(\Delta';\Omega')$ such that $\Delta \subseteq \Delta'$, we have that $\Box\alpha \in \Delta'$ iff $\Box\alpha \in \Delta$ and $\sim\Box\alpha \in \Delta'$ iff $\sim\Box\alpha \in \Delta$.

Proof. Follows from the fact that a root-theory is saturated and our logics contains schema A15; so, by construction, for any $\alpha \in \mathcal{L}_m$, either $\Box\alpha \in \Delta$ or $\sim\Box\alpha \in \Delta$.

Lemma 9 Given a language $\mathcal{L}_m$, a modal logic L and a root-theory $0 = (\Delta_0;\Omega_0)$, let $(\Delta;\Omega)$ be a 0-theory in $\mathcal{L}_m$, with $\alpha \to \beta = \gamma \in \Omega$. Then there exists a 0-theory $(\Delta';\Omega')$ in the same language, with $\alpha \in \Delta'$, $\beta \in \Omega'$, $\Delta \subseteq \Delta'$.

Proof. From the hypothesis follows that $([\Delta,\alpha];\beta)$ is a consistent k-theory of constant domain in $\mathcal{L}_m$ ([Gab74], lemma 3).

We assume that $\alpha_1, \alpha_2, \ldots$ is an enumeration of the formulæ in $\mathcal{L}_m$.

We define inductively a sequence of k-theories $(\Delta_n;\Omega_n)$ such that for each n, $\Delta_n \subseteq \Delta_{n+1}$ and $\Omega_n \subseteq \Omega_{n+1}$.

Base case: $(\Delta_0;\Omega_0) = ([\Delta,\alpha];\beta)$.

Step case: suppose $\mathtt{p}_n = (\Delta_n;\Omega_n)$ is defined and consistent. We define $(\Delta_{n+1};\Omega_{n+1})$. There are two main cases.

A) $(\Delta_n;[\Omega_n,\alpha_n])$ is consistent.

A1) $\alpha_n = \beta \to \gamma | \Box\beta$; then $\Delta_{n+1} = \Delta_n$ and $\Omega_{n+1} = [\Omega_n,\alpha_n]$.

A2) $\alpha_n = \forall x.\beta(x)$; then, since $\mathtt{p}_n$ has the constant domain property in $\mathcal{L}_m$, there must be in $\mathcal{L}_m$ a variable y such that $\mathtt{p}_n = (\Delta;[\Omega,\alpha_n,\beta(y/x)])$ is consistent; $\mathtt{p}_n$ is of constant domains in $\mathcal{L}_m$ ([Gab74], lemma 2).

B) $(\Delta_n;[\Omega_n,\alpha_n])$ is inconsistent. Then $([\Delta_n,\alpha_n];\Omega_n)$ is consistent, by lemma 6. Let $\Delta_{n+1} = [\Delta_n,\alpha_n]$ and $\Omega_{n+1} = \Omega_n$.

Let $\Delta' = \bigcup_{n\in N}\Delta_n$ and $\Omega' = \bigcup_{n\in N}\Omega_n$; then, using lemma 8, it follows that (Δ',Ω') is a 0-theory in $\mathcal{L}_m$.

We can now prove the following (similar to [Gab81], lemma 3.4.3).

Lemma 10 Given a language $\mathcal{L}_m$ and a modal logic L, let 0 be a root-theory, and $(\Delta;\Omega)$ be a 0-theory; then $(\Delta;\Omega)$ can be extended to a terminal 0-theory $(\Delta';\Omega')$ in the same language.

Proof. Let $\gamma = \forall x.x \vee \sim x$; since $\vdash_L \approx\gamma$ (by lemma 3), it must be $\gamma \to \bot \in \Omega$. By lemma 9, there exists a 0-theory $\mathtt{t} = (\Delta';\Omega')$

in $\mathcal{L}_m$, such that $[\Delta, \gamma] \subseteq \Delta'$ and $\Omega \subseteq \Omega'$; for every formula α in $\mathcal{L}_m$, $\alpha \vee \sim \alpha \in \Delta'$, as $\gamma \vdash_L \alpha \vee \sim \alpha$ and $\mathtt{t}$ is saturated; but then, again by saturation of $\mathtt{t}$, either $\alpha \in \Delta'$ or $\sim \alpha \in \Delta'$.

We can now move on to the proof of the main theorem.

Thm. 2 (Soundness and completeness for N*2At*).
For any formula α, $\vdash_{N2At} \alpha$ iff, for every N*2At*-model $\mathcal{M}$, $\vDash_M \alpha$.

Proof. *Left to right.* It is routine to check that the axioms are valid, and that the inference rules are validity-preserving in every model defined according to def.4.

Right to left. The idea is that of showing, given a non-theorem, how to build a counter-model, where the minimum is a root-theory 0, and the other elements of the frame are 0-theories; the interpretation is the canonical one (each propositional letter is interpreted as itself).

Assume $\nvdash_{N2At} \alpha$; then, there must be a consistent k-theory (Δ, Ω) in $\mathcal{L}_m$, with $\alpha \in \Omega$, and consequently by lemma 7, also a root-theory $0 = (\Delta_0, \Omega_0)$ which extends (Δ, Ω), in an extended language $\mathcal{L}'_m$ wrt N*2At*. Then a counter-model $\mathcal{K} = (\mathcal{S}, \mathcal{R}, \rho)$ for α in $\mathcal{L}'_m$, with $\mathcal{S} = (S, \leq, \mathbf{0})$, can be built as follows.

Let S be the set of all the 0-theories (Δ', Ω') such that $\Delta \subseteq \Delta'$; let $(\Delta', \Omega') \leq (\Delta''; \Omega'')$ iff $\Delta' \subseteq \Delta''$; let $\mathbf{0} = (\Delta_0; \Omega_0)$. For any variable $x \in \mathcal{L}'_m$, $\rho(x) = \{(\Delta; \Omega) : x \in \Delta\}$. Let $\mathcal{R} = \{X : X = \{(\Delta; \Omega) : \alpha \in \mathcal{L}'_m \ \& \ \alpha \in \Delta\}\}$.

(A) $\mathcal{S}$ is a *2At*-frame.

Proof. (a) By construction, $(S, \leq)$ is a partial order and has a minimum.
(b) $\mathcal{S}$ satisfies condition *mh*. In fact, by lemma 10 we know that for each $\mathtt{a} \in S$ there is a terminal $\mathtt{b} \in S$ such that $\mathtt{a} \leq \mathtt{b}$.
(c) $\mathcal{S}$ satisfies condition *fc*. In fact, since all the instances of schema A6 are in Δ, for any formula β in $\mathcal{L}'_m$ there must be a variable x such that $\|\beta\|_\rho = \|x\|_\rho$ (by saturation).

(B) $\mathcal{K}$ is a N*2At*-model.

In order to prove this, we still need to show that the canonical interpretation can be extended to all the formulae, as follows.

(*) Given $(\Delta; \Omega) \in S$, for any formula γ in $\mathcal{L}'_m$, $(\Delta; \Omega) \in \|\alpha\|_\rho$ iff $\gamma \in \Delta$.

Proof. Both halves (A and B) are by induction on complexity of the formulæ.

A) Assume $(\Delta;\Omega) \in \|\gamma\|_\rho$, to prove $\gamma \in \Delta$.

A1) $\gamma = \alpha \to \beta$. If $\gamma \notin \Delta$, then $\gamma \in \Omega$, since any 0-theory is complete. Then, by lemma 9, there exists a 0-theory $(\Delta';\Omega')$ in the same language, with $\alpha \in \Delta'$, $\beta \in \Omega'$, $\Delta \subseteq \Delta'$. We can apply the induction hyp.; so $(\Delta;\Omega) \leq (\Delta';\Omega')$, $(\Delta';\Omega') \vDash_M \alpha$ and $(\Delta';\Omega') \nvDash_M \beta$; given the interpretation rule for $\to$ in def. 4, this is not compatible with $(\Delta;\Omega) \in \|\gamma\|_\rho$.

A2) $\gamma = \forall x.\alpha(x)$. Since $(\Delta;\Omega)$ is of constant domain, if $\gamma \in \Omega$, then, for some variable $y \in \mathcal{L}_m$, $\alpha(y/x) \in \Omega$. Applying the induction hyp. and the interpretation rule for $\forall$, we get a contradiction.

A3) $\gamma = \Box\alpha$. Since $(\Delta;\Omega)$ is a 0-theory, if $\gamma \in \Omega$, then $\alpha \notin \Delta_0$. Applying the induction hyp., $(\Delta_0;\Omega_0) \nvDash_M \alpha$, and then, applying the interpretation rule for $\Box$, $(\Delta;\Omega) \nvDash_M \Box\alpha$, in contradiction with the hypothesis.

B) Assume $\gamma \in \Delta$, to prove $(\Delta;\Omega) \in \|\alpha\|_\rho$.

B1) $\gamma = \alpha \to \beta$. If $(\Delta;\Omega) \notin \|\gamma\|_\rho$ then, by def. of interpretation, there exists $(\Delta';\Omega') \in W$ such that $(\Delta;\Omega) \leq (\Delta';\Omega')$, (ie, $\Delta \subseteq \Delta'$) with $(\Delta';\Omega') \vDash \alpha$ and $(\Delta';\Omega') \nvDash \beta$; so, by A (first half of the proof), $\alpha \in \Delta'$ and, by induction hyp., $\beta \notin \Delta'$; it follows saturation of the theory, since $\gamma \in \Delta'$ by hyp., that $\beta \in \Delta'$, a contradiction.

B2) $\gamma = \forall x.\alpha$. If $(\Delta;\Omega) \notin \|\gamma\|_\rho$ then, applying the interpretation rule for $\forall$, $(\Delta;\Omega) \notin \|\alpha(y/x)\|_\rho$ for some $y \in \mathcal{L}_m$, then, by induction hyp., $\alpha(y/x) \notin \Delta$; a contradiction follows.

B3) $\gamma = \Box\alpha$. If $(\Delta;\Omega) \notin \|\gamma\|_\rho$ then, by def. of interpretation, $(\Delta_0;\Omega_0) \notin \|\alpha\|_\rho$; then, by induction hyp., $\alpha \notin \Delta_0$. Since $(\Delta;\Omega)$ is a 0-theory, it follows $\Box\alpha \notin \Delta$.

□

This proof can be modified in order to get completeness also for *2At* and N*C2h*.

Thm. 3 (Soundness and completeness for *2At*).
For any formula α, $\vdash_{2At} \alpha$ iff, for every *2At*-model $\mathcal{M}$, $\vDash_M \alpha$.

Proof. The proof of theorem 2 can be modified omitting all the aspects related to modality. So, the canonical model is just a set of ck-theories in a non-modal language $\mathcal{L}$.

Thm. 4 (Soundness and completeness for N$C2h$).
For any formula α, $\vdash_{NC2h} \alpha$ iff, for every N$C2h$-model $\mathcal{M}$, $\models_M \alpha$.

Proof. The proof of theorem 2 can be modified omitting all the aspects related to terminability (so, lemma 10 is not used).

4 Topology and Kripke models

It is possible to look at Kripke models from a topological point of view, referring to a class of topologies that can be presented in several different, equivalent ways. We first introduce some standard topological notions [RS63].

In general, a *topological space* $\mathcal{S}$ is given as a pair $(S, \mathcal{O})$ where S is a set of points and $\mathcal{O}$ is the class of the *open sets* (also *opens*) of S. Set-theoretic complements of open sets are *closed sets.* The opens can also be presented algebraically, as the elements that satisfy $X = \mathsf{I}X$ in a complete Boolean algebra, isomorphic to $(\wp(S), \bigcap, \bigcup, \cap, \cup, -)$, where I is added as an interior operator defined by the Kuratowski axioms ([RS63], chp. 3); this means that $\varnothing, S \in \mathcal{O}$, and that $\mathcal{O}$ is closed with respect to arbitrary unions ($\bigcup$), and finite intersections ($\cap$). A closure operator C can be defined as the dual of I. A subset $\mathcal{B} \subseteq \mathcal{O}$ forms a *basis* for the topology iff every open set can be represented as a union of elements of $\mathcal{B}$. Given $A \in \mathcal{O}$, we write $\mathcal{S}^A$ for the *restriction* of $\mathcal{S}$ to A.

An operation of *pseudo-complement*, or Heyting complement, can be defined, for open sets, as the interior of the set-theoretic complement, ie $A^* = \mathsf{I}(-A)$. This notion can be generalised, for $A, B \in \mathcal{O}$, to that of *relative pseudo-complement* $A{\Rightarrow}B$, as follows.

Lemma 11 [Tar56]. Let $A{\Rightarrow}B = \mathsf{I}(-A \cup B)$ in $\mathcal{S}$ as above. Then, for any $A, B, C \in \mathcal{O}$:
(a) $A^* = A \Rightarrow \varnothing$
(b) $C \subseteq A{\Rightarrow}B$ iff $C \cap A \subseteq B$.

Equivalence b is often used as definition of $\Rightarrow$; intuitively, it states that $A{\Rightarrow}B$ gives the largest open subspace in which A is set-theoretically included in B.

A subset of S is *regular open* when it is equivalent to the interior of its closure; since $\mathsf{I}(\mathsf{C}A)) = A^{**}$, however, it turns out more convenient for us to intend *regularisation* as double pseudo-complement.

The notion of open set together with pseudo-complement and regularisation offer to the intuition a qualitatively significant way to partition points in a space. For $A \in \mathcal{O}$ and $\mathtt{p} \in S$, in fact, we can say that $\mathtt{p}$ is an *internal* point of A iff $\mathtt{p} \in A$; that $\mathtt{p}$ is an *external* point of A iff $\mathtt{p} \in A^*$; that $\mathtt{p}$ is an *internal boundary* point of A iff $\mathtt{p} \notin A$ and $\mathtt{p} \cup A^{**}$; that $\mathtt{p}$ is an *external boundary* point of A iff $\mathtt{p} \notin A$ and $\mathtt{p} \notin \cup A^{**}$. The presence of boundary points will allow us introducing, further on, different notions of connection relation.

Def. 7 Given a space $\mathcal{S} = (S, \mathcal{O})$ we can always define a *specification order* $\sqsubseteq$ on $\mathcal{S}$, s.t. $\mathtt{p} \sqsubseteq \mathtt{q}$ iff, for any $A \in \mathcal{O}$, $\mathtt{p} \in A$ implies $\mathtt{q} \in A$.

Def. 8 Let $(S, \mathcal{O})$ be a space.
We say that an open set is *prime* (strongly compact, in [RS63]) iff, whenever it is included in a union of open sets, it is included in one of them.
We say that an open set is *minimal* (or atomic) iff there does not exist a non-empty open set which is properly included in it.
We say an open set is *terminable* iff every open subset of that set includes a non-empty minimal open set. We can see that, if the space is terminable, for every point $\mathtt{p}$, either $\mathtt{p}$ is internal to a minimal set, or is a boundary point of some minimal set.

Def. 9 A space $(S, \mathcal{O})$ is *Alexandroff* iff $\mathcal{O}$ is also closed wrt arbitrary intersection ($\bigcap$).

Def. 10 Given a pre-order $(S, \leq)$, we call $(S, \mathcal{O})$ the *order topology* determined by $\leq$ on S, iff $\mathcal{O} = \mathcal{U}_{\leq}$, ie the open sets are the upper-closed sets wrt $\leq$. For $\mathtt{a} \in S$, $\{\mathtt{a} \uparrow\}$ is the smallest upper-closed set that contains $\mathtt{a}$ (ie, the upper-closed generated by $\mathtt{a}$.)

Def. 11 A topological space is T_0 (has the T_0 separation property) iff for any two points, there is an open set containing one of them and not the other one.

Lemma 12 Thm.9, chp.3 [Gab81]. If $(S, \leq)$ is a partially ordered set, then the order topology $\mathcal{S}_{\leq}$ determined by $\leq$ on S is an Alexandroff, T_0 space, and the specification order on $\mathcal{S}_{\leq}$ is isomorphic to $\leq$.
If $\mathcal{S} = (S, \mathcal{O})$ is an Alexandroff, T_0 space then $(S, \sqsubseteq)$ is a partially ordered set, and $\mathcal{S}$ is isomorphic to the order topology determined by $\sqsubseteq$ on S.

Lemma 13 $(S, \mathcal{O})$ is an order topology iff the class of its prime subsets form a basis for it [FS79].

From lemma 12 follows that Alexandroff spaces and order topologies are equivalent notions.

Lemma 13 intuitively says that a space is Alexandroff iff it has a canonical basis which is minimal, since it is made of elements that cannot be represented as union of other elements. So, each open set has a unique decomposition in terms of basic elements. This, if not sufficient, still fits in quite naturally when we want to model digital representation.

Lemma 14 Let $\mathcal{S}$ be an Alexandroff space. Then, for $A, B \in \mathcal{O}$, $A \Rightarrow B$ is the union of all the prime $X \in \mathcal{O}$ s.t. $X \cap A \subseteq B$.

Proof. By lemmas 11 and 13.

4.1 Topological interpretation

Intuitively, given a Kripke model $\mathcal{M} = ((S, \leq, \mathbf{0}), \mathcal{R}, \rho)$, the truth-sets generated by the interpretation of the formulæ ($\|\alpha\|_\rho$) are the regions; that is, the elements of $\mathcal{R}$ are the sets of points that satisfy some formula. Each point, on the other hand, can be defined by the set of the regions that have it as an element (essentially, this is the strategy for the completeness theorem).

Lemma 15 There is a one-to-one correspondence between Kripke *C2h*-frames and T_0, prime, Alexandroff spaces.
Similarly, there is a one-to-one correspondence between Kripke *2At*-frames and T_0, prime, terminable, Alexandroff spaces.

Proof. Essentially a corollary of 12 and def. 3.
The fact that the space is prime corresponds, under the T_0 restriction, to the fact that the frame has a minimum.
Terminability corresponds to the condition *mh*.

We now can introduce the following notions.

Def. 12 Let $(S, \mathcal{O})$ be an Alexandroff space; let $\mathcal{A} \subseteq \mathcal{O}$; let $\mathcal{B}$ be the smallest subset of $\mathcal{O}$ such that $\mathcal{A} \subseteq \mathcal{B}$, if $X, Y \in \mathcal{A}$ then $X \Rightarrow Y \in \mathcal{B}$, and, if $F \in (\mathcal{A} \mapsto \mathcal{B})$, $\bigcap\{F(Y) : Y \in \mathcal{A}\} \in \mathcal{B}$. Then, we say that $\mathcal{A}$ is a *r-set* in $(S, \mathcal{O})$ iff $\mathcal{B} \subseteq \mathcal{A}$.

Def. 13 We say here that $(S, \mathcal{O}, \mathcal{R})$ is a *region space* (or a *r-space* for short) whenever $(S, \mathcal{O})$ is an Alexandroff space and $\mathcal{R} \subseteq \mathcal{O}$ is a r-set in $(S, \mathcal{O})$.

We will say that a r-space $(S, \mathcal{O}, \mathcal{R})$ has a topological property whenever the space $(S, \mathcal{O})$ has it; if the property refers to $\mathcal{R}$, we may also say that the space has that property wrt $\mathcal{R}$. Besides, a notion of restriction can be defined also for r-spaces, as follows.

Def. 14 Let $A\mathcal{O}$; the restriction of $\mathcal{S}_r$ to A is $\mathcal{S}_r^A = (A, \{X \cap A : X \in \mathcal{O}\}, \{Y \cap A : Y \in \mathcal{R}\})$.

It is routine to check that $\mathcal{S}_r^A$ is a r-space.

Lemma 16 Given a Kripke L-model $\mathcal{M} = ((S, \leq, \mathbf{0}), \mathcal{R}, \rho)$, let the *topological interpretation* of each formula be its truth-set wrt to ρ, ie $\|\alpha\|_\rho$.
Then $(S, \mathcal{U}_\leq, \mathcal{R})$ is an r-space.
Besides, for any $\alpha \in \mathit{Wff}$, the topological interpretation always satisfies the following equalities (some of these give a representation for intuitively significant relations, as indicated).

1. $\|\alpha \to \beta\|_\rho = \|\alpha\|_\rho \Rightarrow \|\beta\|_\rho$
2. $\|\forall x.\alpha(x)\|_\rho = \bigcap\{\|\alpha(y/x)\|_\rho : y \in \mathit{Var}\}$
3. $\|\Box\alpha\|_\rho = S$ iff $\|\alpha\|_\rho = S$
4. $\|\Box\alpha\|_\rho = \varnothing$ iff $\|\alpha\|_\rho \neq S$
5. $\|\bot\|_\rho = \varnothing$
6. $\|\top\|_\rho = S$
7. $\|\alpha \wedge \beta\|_\rho = \|\alpha\|_\rho \cap \|\beta\|_\rho$
8. $\|\alpha \vee \beta\|_\rho = \|\alpha\|_\rho \cup \|\beta\|_\rho$
9. $\|\exists x.\alpha(x)\|_\rho = \bigcup\{\|\alpha(y/x)\|_\rho : y \in \mathit{Var}\}$
10. $\| \sim \alpha\|_\rho = \|\alpha\|_\rho^*$ (pseudo-complement)
11. $\|{\approx}\alpha\|_\rho = \|\alpha\|_\rho^{**}$ (regularisation).
12. $\|\alpha \leftrightarrow \beta\|_\rho = \|\alpha\|_\rho \Leftrightarrow \|\beta\|_\rho$ (equivalence).
13. $\mathtt{a} \in \|\alpha\|_\rho, \mathtt{a} \leq \mathtt{b}$ implies $\mathtt{b} \in \|\alpha\|_\rho$
14. $\mathbf{0} \in \|\alpha\|_\rho$ iff $\|\alpha\|_\rho = S$
15. $\|\alpha \to \beta\|_\rho = S$ iff $\|\alpha\|_\rho \subseteq \|\beta\|_\rho$ (inclusion or part).
16. $\| \Diamond \alpha\|_\rho = S$ iff $\|\alpha\|_\rho \neq \varnothing$ (non-emptiness).
17. $\| \sim(\alpha \wedge \beta)\|_\rho = S$ iff $\|\alpha\|_\rho \cap \|\beta\|_\rho = \varnothing$ (disjointness).
18. $\| \Diamond(\alpha \wedge \beta)\|_\rho = S$ iff $\|\alpha\|_\rho \cap \|\beta\|_\rho \neq \varnothing$ (overlapping).

Proof. By lemma 4, every truth-set is an upper-closed set wrt $\leq$, ad so $\mathcal{R}$ is a subclass of the open sets in the order topology $(S,\mathcal{U}_{\leq})$. Besides, given the interpretation rules of $\rightarrow, \forall$ and the condition *fc* in def. 4, $\mathcal{R}$ must be closed in the sense of def. 12.
All the equivalences are proved by the properties of Kripke models (def. 4, lemma 5), and by those of the topological operators.

Lemma 17 Let $(S, \mathcal{O}, \mathcal{R})$ be a T_0, prime r-space, and $\mathtt{p} \in S$ a point s.t. for all $X \in \mathcal{O}$, $X \neq S$, $\mathtt{p} \notin X$. Let $\rho : \mathit{Var} \mapsto \mathcal{R}$ be a surjective function from the variables of the language into $\mathcal{R}$. Then $\mathcal{M} = ((S, \sqsubseteq, \mathtt{p}), \mathcal{R}, \rho)$ is a model (for N*C2h* if the language is modal; otherwise for *C2h*; if the r-space is terminable, respectively for N*2At* or *2At*).

Proof. By lemma 15, $(S, \sqsubseteq, \mathtt{p})$ gives a frame. By the properties of r-sets (def. 12), ρ can be extended to an interpretation for all formulæ in $\mathcal{M}$ as in def. 4.

5 Topological relations

We now will consider how some qualitatively significant notions can be expressed using topology. As a first example, we can define the property of an open set's being connected, and so, intuitively, made of one piece, by noting that a connected open cannot be 'divided' by the boundary of any open set. The following is just a variant of the standard definition of a connected set as one that is not equal to the sum of two disjoint non-empty opens.

Def. 15 Let $\mathcal{S} = (S, \mathcal{O})$ be a topological space. $A \in \mathcal{O}$ is connected in $\mathcal{S}$ iff, for every $X \in \mathcal{O}$, $A \subseteq X \cup X^*$ implies $A \subseteq X$ or $A \subseteq X^*$.

In the following, let $\mathcal{S} = (S, \mathcal{O})$ be Alexandroff space, and $\mathcal{S}_r = (S, \mathcal{O}, \mathcal{R})$ be a prime r-space. We can now introduce an idea of relativisation wrt the expressive power of the language, represented by the collection $\mathcal{R}$ of the regions. So for connectedness and the following.

Def. 16 $A \in \mathcal{R}$ is $\mathcal{R}$-connected in $\mathcal{S}_r$ (or r-connected in $\mathcal{S}$ wrt $\mathcal{R}$) iff, for every $X \in \mathcal{R}$, $A \subseteq X \cup X^*$ implies $A \subseteq X$ or $A \subseteq X^*$.

Whenever $\mathcal{S}_r$ is prime, the metatheoretical statement $A \subseteq X$ *or* $A \subseteq X^*$ is equivalent to a theoretical one, $(A \Rightarrow X) \cup (A \Rightarrow X^*) = S$.

5.1 Connectedness with restrictions

A relation that in prime r-spaces is stronger than $\mathcal{R}$-connectedness can be introduced as follows.

Def. 17 Given $T \in \mathcal{O}$, $A \in \mathcal{R}$ is *strongly $\mathcal{R}$-connected* in $\mathcal{S}_r^T$ ($T \subseteq \mathsf{SC}(A)$) iff, for every prime $V \subseteq T$, the intersection $A \cap T$ is $\mathcal{R}$-connected in $\mathcal{S}_r$.

Strong $\mathcal{R}$-connectedness can also be considered from another point of view, though, introducing the following.

Def. 18 For $A, B \in \mathcal{R}$, $T \in \mathcal{O}$, we say that B *nowhere $\mathcal{R}$-splits* A in $\mathcal{S}_r^T$ ($T \subseteq \mathsf{NS}(A, B)$) iff, for every prime $V \subseteq T$, we have that $V \cap A \subseteq B \cup B^*$ implies $V \cap A \subseteq B$ or $V \cap A \subseteq B^*$.

Lemma 18 Given $T \in \mathcal{O}$, $T \subseteq \mathsf{SC}(A)$ iff, for every $X \in \mathcal{R}$, $T \subseteq \mathsf{NS}(A, X)$ (ie, iff $T \subseteq \bigcap_{X \in \mathcal{R}} \mathsf{NS}(A, X)$).

Proof. By def.17, def.16, def.18 and def. 14.

Differently from $\mathcal{R}$-connectedness, strong $\mathcal{R}$-connectedness can be expressed in a logical languages at the object level, without using the modality (lemma 27).

We will now show how imposing some constraints on the r-spaces, we can make the property of strong r-connectedness coincide with r-connectedness, at least for the regular regions.

Def. 19 We say that an open subspace T is *$\mathcal{R}$-trivial* in $\mathcal{S}$ iff for every regular $A \in \mathcal{R}$, $T \subseteq \mathsf{SC}(A)$; otherwise, we say that it is *$\mathcal{R}$-nontrivial*.

Def. 20 We say that $\mathcal{S}$ is *$\mathcal{R}$-disjunctive* iff for every $A \in \mathcal{R}$, either $A = S$, or A is $\mathcal{R}$-trivial.

Here we get the main lemma.

Lemma 19 In a prime $\mathcal{R}$-disjunctive r-space $\mathcal{S}_r = (S, \mathcal{O}, \mathcal{R})$, for any regular $A \in \mathcal{R}$, A is $\mathcal{R}$-connected iff it is strongly $\mathcal{R}$-connected.

Proof. *Left to right.* Straightforward; from the definition, in any prime space, strong $\mathcal{R}$-connectedness implies $\mathcal{R}$-connectedness. *Right to left.* If $A = A^{**} \in \mathcal{R}$ is not strongly $\mathcal{R}$-connected in $\mathcal{S}_r$ then, for some $B \in \mathcal{R}$, there must be a region $F = A \Rightarrow B \cup B^*$ such that $A \Rightarrow B \cup B^* \not\subseteq (A \Rightarrow B) \cup (A \Rightarrow B^*)$; then F is not trivial, since $F \not\subseteq \mathsf{SC}(A)$; but since $\mathcal{S}_r$ is $\mathcal{R}$-disjunctive, by definition, it must be then $F = S$, and so A is not $\mathcal{R}$-connected in $\mathcal{S}_r$.

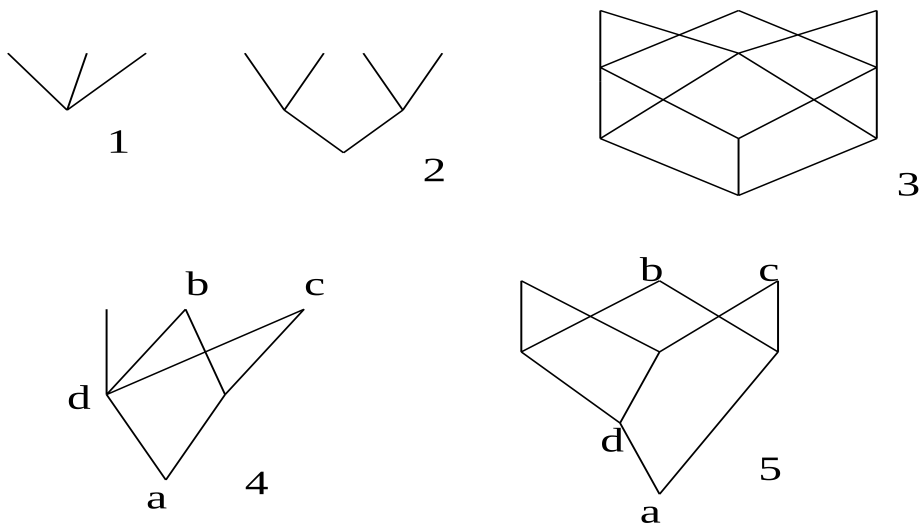

Figure 1:

Fig. 1 shows three *2At*-frames (1,2,3), s.t. the principal models defined on them are $\mathcal{R}$-disjunctive, whereas this is not the case for frames (4,5). In both 4 and 5 the set $(\{b\} \cup \{c\})^{**}$ is connected at a, but fails to be connected at d.

5.2 Connection and disconnection

We have two different ways of defining the relations of connection and disconnection (here also: interconnection and interdisconnection, in order to avoid confusion with unary connectedness, aka self-connectedness).

Intuitively, two overlapping regions must have as a common part some non-empty subregion. On the other hand, interconnected regions either may be overlapping, or may be simply adjacent one to the other; in the latter case, they will share some points of their boundaries, without overlapping. This situation cannot be modelled directly in our language, since we cannot express independently a notion of boundary. However, we can rely on the fact that, regularising the union of two regions that share some boundary interval (not just isolated boundary points) we get a region that is strictly larger than the simple union; in other terms, whenever non-empty regions A and B are "firmly" adjacent, $A^{**} \cup B^{**} \neq (A \cup B)^{**}$.

Alternatively, interconnection can be defined referring to connectedness; two regions, not necessarily self-connected ones, are interconnected whenever they include non-empty, self-connected parts such that their regularised union is another self-connected region.

This second definition is based on an existential statement, and so depends on $\mathcal{R}$. On the contrary, the first definition does not depend on $\mathcal{R}$, but requires modality to be encoded in our logic (as we shall see).

We can make these distinctions more precise with the following. Let $\mathcal{S}$ and $\mathcal{S}_r$ be as before.

Def. 21 Let $\mathsf{DC}(A,B) = (A \cap B)^* \cap ((A \cup B)^{**} \Rightarrow (A^{**} \cup B^{**}))$.
We say that two regions A, B are *interdisconnected* in $\mathcal{S}$ whenever $\mathsf{DC}(A,B) = S$.
Otherwise, we say that they are *interconnected*.

Def. 22 We say that two regions A, B are $\mathcal{R}$-*interconnected* in $\mathcal{S}_r$ ($\mathsf{RC}(A,B) = S$) iff there exist not-empty $X, Y \in \mathcal{R}$ s.t. $X \subseteq A, Y \subseteq B$ and $(X \cup Y)^{**}$ is $\mathcal{R}$-connected in S_r.
Otherwise, they are said to be $\mathcal{R}$-*interdisconnected*.

The following intuitively justifies our definition of $\mathcal{R}$-interconnection.

Lemma 20 For any regular $A \in \mathcal{O}$, A is $\mathcal{R}$-connected in $\mathcal{S}_r$ iff there do not exist non-empty $B, C \in \mathcal{R}$ which are interdisconnected such that $A \subseteq B \cup C$, $A \cap B \neq \varnothing, A \cap C \neq \varnothing$.

Proof. *Left to right.* If A can be split in two such regions B, C, it cannot be $\mathcal{R}$-connected, as follows from def. 16.
Right to left. Assume A is not $\mathcal{R}$-connected. Then, there is some $X \in \mathcal{R}$ such that $A \Rightarrow X \cup X^* = S$, $A \Rightarrow X \neq S$, $A \Rightarrow X^* \neq S$. Now assume that $(A \cap X)$ and $(A \cap X^*)$ are not interdisconnected. Then, by def. 21 there must be a point $\mathtt{b} \in S$ such that $\mathtt{b} \in (A \cap (X \cup X^*))^{**}$ and $\mathtt{b} \notin (A \cap X)^{**}$, $\mathtt{b} \notin (A \cap X^*)^{**}$. So $\mathtt{b} \in A$, $\mathtt{b} \notin (X \cup X^*)^{**}$. But then, $A \Rightarrow X \cup X^* \neq S$, contrary to the assumption.

Lemma 21 For any regular, non-empty $\mathcal{R}$-connected $A, B \in \mathcal{R}$, $(A \cup B)^{**}$ is $\mathcal{R}$-connected in $\mathcal{S}$ iff A and B are interconnected.

Proof. *Left to right.* From lemma 20. *Right to left.* If A and B are interconnected, they share a boundary interval; it follows that $(A \cup B)^{**}$ is connected, and then also $\mathcal{R}$-connected.

5.3 Well-connectedness

We can now consider an additional restriction on our spaces.

Def. 23 We say that a r-space $(S, \mathcal{O}, \mathcal{R})$ is *well-connected* whenever:
(a) for any $A, B, C \in \mathcal{R}$, if A and C are interdisconnected and B and C are interdisconnected, then $A \cup B$ and C are interdisconnected;
(b) for any $A \in \mathcal{R}$, A equals the union of its regular, $\mathcal{R}$-connected subregions.

Lemma 22 In any well-connected, prime r-space $(S, \mathcal{O}, \mathcal{R})$, for any non-empty $A, B \in \mathcal{R}$:
A) A and B are interdisconnected, iff they are $\mathcal{R}$-interdisconnected.
B) A and B are interconnected, iff they are $\mathcal{R}$-interconnected.

Proof. A) *Left to right.* If A and B were $\mathcal{R}$-interconnected, there should be non-empty $C \subseteq A$, $D \subseteq B$ s.t. their regular union is $\mathcal{R}$-connected; but, since A and B are interdisconnected, $A^{**} \cup B^{**} = (A \cup B)^{**}$, and $A^{**} \cup B^{**} \subseteq A^{**} \cup A^{*}$. $(C \cup D)^{**} \subseteq A^{**} \cup B^{**}$, so $(C \cup D)^{**} \subseteq A \cup A^{*}$; however, $(C \cup D)^{**} \nsubseteq A$, $(C \cup D)^{**} \nsubseteq A^{*}$, contradicting the $\mathcal{R}$-connectedness of $(C \cup D)^{**}$.
Right to left. Since A and B are not $\mathcal{R}$-interconnected, for every non-empty, regular, $\mathcal{R}$-connected C, D s.t. $C \subseteq A$, $D \subseteq B$, we have that $(C \cup D)^{**}$ is not $\mathcal{R}$-connected, so, by lemma 21, C and D must be interdisconnected. The union of all the non-empty, $\mathcal{R}$-connected, regular open subsets of A equals A (by property b of def. 23; and similarly for B. Then, using property a of def.23, A and B must be interdisconnected.
B) Consequence of part A and the definitions.

The following shows that $\mathcal{R}$-interconnection has a significant property that, according to [PS98], as discussed in the introduction, connection on regions should satisfy.

Lemma 23 In any well-connected, prime r-space $(S, \mathcal{O}, \mathcal{R})$, for any $A, B, C \in \mathcal{R}$, if $A \cup B$ and C are $\mathcal{R}$-interconnected, then either A and C or B and C are $\mathcal{R}$-interconnected.

Proof. By lemma 22 and property a of def.23.

In fig. 2 the directed graph on the right is a *2At*-frame where the principal model is $\mathcal{R}$-nontrivial and well-connected; this particular one can be associated in a natural way to the 2-d representation on the

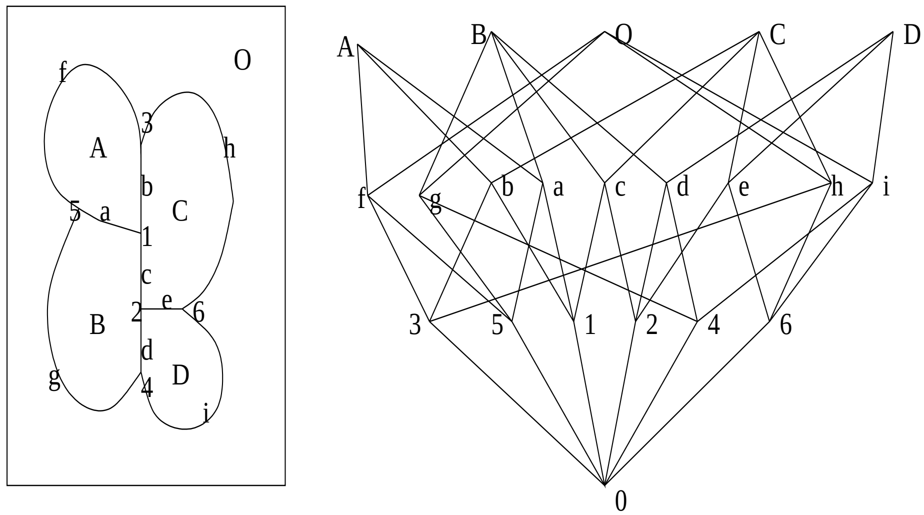

Figure 2:

left. By contrast, in fig. 1, examples 1 and 2 give principal models that are not well-connected, whereas example 3 gives one that is $\mathcal{R}$-trivial.

Fig. 3 shows an example of *2At*-frame where the principal model is not well-connected; taking A as the set of the even points and B as the set of the odd points, both A and B are interdisconnected with L (limit region arising out of an infinite intersection), whereas $A \cup B$ is not.

5.4 Tangential and non-tangential parts

It is now possible to make some distinctions within the part-whole relation, along the lines of RCC [CBGG97]. Let $\mathcal{S}_r = (S, \mathcal{O}, \mathcal{R})$ be a prime r-space.

Def. 24 In $\mathcal{S}_r$, for any $A, B \in \mathcal{O}$, we say that A is a *non-tangential part* ($\mathcal{R}$-nontangential part) of B iff A is part of B and is interdisconnected ($\mathcal{R}$-interdisconnected) with B^*.

In particular, A is a non-tangential part of B^{**} iff A is interdisconnected with B^* (and similarly for the corresponding $\mathcal{R}$-relations).

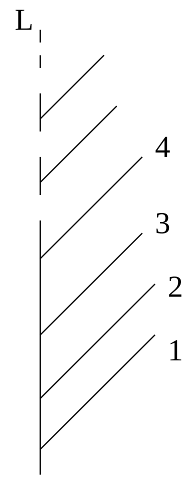

Figure 3:

Def. 25 In $\mathcal{S}_r$, for any $A, B \in \mathcal{O}$, we say that A is a *tangential part* ($\mathcal{R}$-tangential part) of B iff $A \subseteq B$ and A is interconnected ($\mathcal{R}$-interconnected) with B.

Immediately from the definitions we get the following.

Lemma 24 In $\mathcal{S}_r$, for any $A, B \in \mathcal{R}$ s.t. $A \subseteq B$, A is ($\mathcal{R}$-)non-tangential part of B iff A is not ($\mathcal{R}$-)tangential part of B.

Lemma 25 If $\mathcal{S}_r$ is well-connected, for any $A, B \in \mathcal{R}$ s.t. $A \subseteq B$, A is non-tangential part of B iff A is not $\mathcal{R}$-tangential part of B.

Proof. From the definitions and lemma 22.

6 Spatial extensions

The principal aim is yet to consider the logics that can be obtained, as axiomatic extensions, by adding either to *2At*, *C2h*, N*2At*, N*C2h* some definitions and axiom schemas that are relevant from the point of view of spatial expressiveness.

Def. 26 $\mathsf{sc}(\alpha) = \forall x.(\alpha \to x \vee \sim x) \to (\alpha \to x) \vee (\alpha \to \sim x)$

$\mathsf{sc}'(\alpha) = \forall x.\Box(\alpha \to x \vee \sim x) \to (\alpha \to x) \vee (\alpha \to \sim x).$

$\mathsf{ext}(\alpha) = \sim \alpha \to K$

$\mathsf{dc}(\alpha, \beta) = \sim(\alpha \wedge \beta) \wedge (\approx(\alpha \vee \beta) \to \approx\alpha \vee \approx\beta)$

$$\mathsf{rc}(\alpha,\beta) = \sim\Box(\mathsf{dc}(\alpha,\beta))$$
$$\mathsf{rc}_r(\alpha,\beta) = \exists xy.\mathsf{ext}(x)\wedge\mathsf{ext}(y)\wedge(x\to\alpha)\wedge(y\to\beta)\wedge\mathsf{sc}(\approx(x\vee y))$$
$$\mathsf{dc}_r(\alpha,\beta) = \sim\Box(\mathsf{rc}_r(\alpha,\beta))$$
$$\mathsf{ntp}(\alpha,\beta) = (\alpha\to\beta)\wedge\mathsf{dc}(\alpha,\sim\beta)$$
$$\mathsf{tp}(\alpha,\beta) = (\alpha\to\beta)\wedge\mathsf{rc}(\alpha,\sim\beta)$$
$$\mathsf{tp}_r(\alpha,\beta) = (\alpha\to\beta)\wedge\mathsf{rc}_r(\alpha,\sim\beta)$$
$$\mathsf{ntp}_r(\alpha,\beta) = (\alpha\to\beta)\wedge\mathsf{dc}_r(\alpha,\sim\beta)$$
$$\mathsf{ov}(\alpha,\beta) = \mathsf{ext}(\alpha\wedge\beta)$$
$$\mathsf{ov}'(\alpha,\beta) = \Diamond(\alpha\wedge\beta)$$
$$K = \forall x.\mathsf{sc}(\approx x)$$
$$\sigma = \alpha\vee(\alpha\to K)$$
$$\tau = \Box K\to\bot$$
$$\theta = \forall xyz.\mathsf{dc}(x,z)\wedge\mathsf{dc}(y,z)\to\mathsf{dc}(x\vee y,z)$$
$$\theta_m = \forall xyz.\Box(\mathsf{dc}(x,z)\wedge\mathsf{dc}(y,z))\to\mathsf{dc}(x\vee y,z)$$
$$\pi = \alpha\to\exists x.(\approx x\to x\wedge\alpha)\wedge\mathsf{sc}(x)\wedge x$$
$$\pi' = \alpha\to\exists x.(\approx x\to x\wedge\alpha)\wedge\mathsf{sc}'(x)\wedge x$$

The extensions we are interested in are such that in them it is possible to express a notion of binary connection (interconnection, rc_r) which satisfies well-connectedness (lemma 31). This can be done in any of the above mentioned logics, introducing definitions of sc (for strong $\mathcal{R}$-connectedness) and dc (for interdisconnection), and adding as axiom schemas σ ($\mathcal{R}$-disjunctive) and θ, π (strongly well-connected).

In a modal logic, sc can be replaced with $\mathsf{sc}'(\alpha)$ (for $\mathcal{R}$-connectedness), σ can be omitted, θ can be replaced with the weaker θ_m and π with π' (well-connected). Non-emptiness can be represented using $\Diamond$.

If the non-modal encoding is embedded in one of the modal logics, adding as an extra axiom schema τ (for non-trivial), we can express non-emptiness with a non-modal operator (ie, *pos*; lemma 29).

7 Logical representation

We can now use the completeness results, and the correspondence between Kripke and topological interpretation (lemmas 16, 17), to show the match between syntactical expressions and topological notions.

In the following, given a model $\mathcal{M} = ((S,\leq,\mathbf{0}),\mathcal{R},\rho)$, we will write $\mathcal{M}'$ for the corresponding r-space $(S,\mathcal{U}_{\leq},\mathcal{R})$. We will also write α' for $\|\alpha\|_\rho$, ie for the region that is the interpretation of α in $\mathcal{M}$.

7.1 Representation for connectedness

Using modality, the fact that a region A is r-connected in a model $\mathcal{M}$, can then be expressed in the language stating the validity in the model of a formula.

Lemma 26 Given a N*2At*-model (N*C2h*-model) $\mathcal{M} = ((S, \leq, \mathbf{0}), \mathcal{R}, \rho)$,
$\models_M \forall x. \Box(\alpha \rightarrow x \vee \sim x) \rightarrow (\alpha \rightarrow x) \vee (\alpha \rightarrow \sim x)$
iff the region α' is $\mathcal{R}$-connected in $\mathcal{M}'$.

Proof. Using the properties of topological interpretation in lemma 16, for any region X, whenever $\alpha' \subseteq X \vee X^*$, we get $(\alpha' \subseteq X) \vee (\alpha' \subseteq X^*)$; then, since a Kripke frame corresponds to a prime space, we get as a conclusion either $\alpha' \subseteq X$ or $\alpha' \subseteq X^*$, corresponding to def. 16.

Differently from $\mathcal{R}$-connectedness, strong $\mathcal{R}$-connectedness can be represented at the object level without using the modality.

Lemma 27 Let ns, sc be as in def.26.
Given a L-model $\mathcal{M} = ((S, \leq, \mathbf{0}), \mathcal{R}, \rho)$, for any $\mathsf{a} \in S$:
(1) $\mathsf{a} \models \mathsf{ns}(\alpha, \beta)$ iff $\{\mathsf{a}\uparrow\} \subseteq \mathsf{NS}(\alpha', \beta')$.
(2) $\mathsf{a} \models \mathsf{sc}(\alpha)$ iff $\{\mathsf{a}\uparrow\} \subseteq \mathsf{SC}(\alpha')$.
(3) $\models_M \mathsf{ns}(\alpha, \beta)$ iff $\mathsf{NS}(\alpha', \beta') = S$.
(4) $\models_M \mathsf{sc}(\alpha)$ iff $\mathsf{SC}(\alpha') = S$.

Proof. (1) Using the interpretation rules in def.4, we get that $\mathsf{a} \models \mathsf{ns}(\alpha, \beta)$ is equivalent to: for every b s.t. $\mathsf{a} \leq \mathsf{b}$, if $\mathsf{b} \models \alpha \rightarrow \beta \vee \sim \beta$ then either $\mathsf{b} \models \alpha \rightarrow \beta$ or $\mathsf{b} \models \alpha \rightarrow \sim \beta$; by the topological interpretation defined in lemma 16 and its properties, def.18 and def. 10 this is equivalent to $\{\mathsf{a}\uparrow\} \subseteq \mathsf{NS}(\alpha', \beta')$.
(2) By lemma 18; then, by the interpretation rule for $\forall$ and part 1 of this lemma, we get equivalence with $\mathsf{a} \models_M \mathsf{sc}(\alpha)$.
(3),(4) Straightforward, taking $\mathsf{a} = \mathbf{0}$.

The following gives the corresponding syntactical conditions for the restrictions that we have introduced in subsection 5.1.

Lemma 28 Let K, σ be as in def.26. Given an L-model $\mathcal{M}$, $\mathcal{M}'$ is $\mathcal{R}$-trivial whenever $\models_M K$.
Besides, $\mathcal{M}'$ is $\mathcal{R}$-disjunctive whenever $\models_M \sigma$.

Proof. By def. 19, def.20, def. 4 and lemma 27.

Now the following can be proved.

Thm. 5 Given a L-model $\mathcal{M}$ s.t. $\models_M \sigma$, for any formula α of the language, $\models_M \mathsf{sc}(\approx\alpha)$ iff α'^{**} is $\mathcal{R}$-connected in $\mathcal{M}'$.

Proof. By lemmas 19 and 27.

With a similar proof, we can show that, if we can semantically rule out $\mathcal{R}$-triviality, it is also possible to express non-emptiness without using modality.

Lemma 29 Let $\mathsf{ext}(\alpha)$ be as in def.26. In any L-model $\mathcal{M}$ s.t. $\mathcal{M}'$ is $\mathcal{R}$-nontrivial, $\models_M \mathsf{ext}(\alpha)$ iff $\alpha' \neq \varnothing$.

However, modality is needed if we want to express syntactically that the model is not trivial. From the interpretation of $\Box$, we immediately get the following.

Lemma 30 Let K, τ be as in def.26. Let $\mathcal{M}$ be a modal L-model. Then $\mathcal{M}'$ is not $\mathcal{R}$-trivial (is $\mathcal{R}$-nontrivial) whenever $\models_M \tau$.

7.2 Representation for other relations

We can now extend the expressiveness result to other relations.

Thm. 6

1. In a L-model $\mathcal{M}$:
   $\models_M \mathsf{dc}(\alpha, \beta)$ iff α' and β' are interdisconnected in $\mathcal{M}'$.
   $\models_M \mathsf{ntp}(\alpha, \beta)$ iff α' is non-tangential part of β' in $\mathcal{M}'$.
2. In a L-model $\mathcal{M}$ s.t. $\models_M \sigma$ (ie, s.t. $\mathcal{M}'$ is $\mathcal{R}$-disjunctive):
   $\models_M \mathsf{rc}_r(\alpha, \beta)$ iff α' and β' are $\mathcal{R}$-interconnected in $\mathcal{M}'$.
   $\models_M \mathsf{tp}_r(\alpha, \beta)$ iff α' is $\mathcal{R}$-tangential part of β' in $\mathcal{M}'$.
3. In a L-model $\mathcal{M}$ s.t. $\models_M \sigma$ and $\models_M \tau$ (ie, s.t. $\mathcal{M}'$ is $\mathcal{R}$-disjunctive and $\mathcal{R}$-nontrivial):
   $\models_M \mathsf{ov}(\alpha, \beta)$ iff α' and β' are overlapping in $\mathcal{M}'$.
4. In a modal L-model $\mathcal{M}$:
   $\models_M \mathsf{rc}(\alpha, \beta)$ iff α' and β' are interconnected in $\mathcal{M}'$.
   $\models_M \mathsf{tp}(\alpha, \beta)$ iff α' is tangential part of β' in $\mathcal{M}'$.
   $\models_M \mathsf{ov}'(\alpha, \beta)$ iff α' and β' are overlapping in $\mathcal{M}$.
5. In a modal L-model $\mathcal{M}$ s.t. $\models_M \sigma$ (ie, s.t. $\mathcal{M}'$ is $\mathcal{R}$-disjunctive):
   $\models_M \mathsf{dc}_r(\alpha, \beta)$ iff α' and β' are $\mathcal{R}$-interdisconnected in $\mathcal{M}'$.
   $\models_M \mathsf{ntp}_r(\alpha, \beta)$ iff α' is $\mathcal{R}$-nontangential part of β' in $\mathcal{M}'$.

Proof. From the definitions, the properties of interpretation (lemma 16, def.4) and the lemmas in section 7.1.

We can characterise well-connectedness syntactically, as follows.

Lemma 31 (A) Let $\mathcal{M}$ be a modal L-model. Then, $\mathcal{M}'$ is well-connected iff $\models_M \theta_m$ and $\models_M \pi'$.
(B) Let $\mathcal{M}$ be a L-model s.t. $\models_M \sigma$. Then, $\mathcal{M}'$ is well-connected if (*strongly* well-connected iff) $\models_M \theta$ and $\models_M \pi$.

Proof. By the definitions in def.26, lemmas 16, 27, 6. Validity of θ, θ_m and π, π' correspond respectively to conditions a and b of def.23.

7.3 JEPD relations

We can now reconsider the expressiveness of the propositional language. Here we see that the Jointly Exhaustive and Pairwise Disjoint (JEPD) sets of relations familiar from the Region-Connection Calculus are expressible in different ways. We are giving some examples.

Thm. 7 Given a modal L-model $\mathcal{M}$ s.t. $\models_M \sigma$, for any negative, non-empty $\alpha, \beta \in \mathit{Wff}$, the following are JEPD:

$\models_M \alpha \leftrightarrow \beta$
$\models_M \mathsf{ov}'(\alpha, \beta) \wedge \mathsf{ov}'(\alpha, \sim\beta)$
$\models_M \mathsf{ntp}_r(\alpha, \beta)$
$\models_M \mathsf{ntp}_r(\beta, \alpha)$
$\models_M \mathsf{tp}_r(\alpha, \beta) \wedge \mathsf{ov}'(\sim\alpha, \beta)$
$\models_M \mathsf{tp}_r(\beta, \alpha) \wedge \mathsf{ov}'(\alpha, \sim\beta)$
$\models_M \mathsf{ntp}_r(\alpha, \sim\beta)$
$\models_M \mathsf{tp}_r(\alpha, \sim\beta)$

Proof. From the representation results in sections 7.1,7.2 and the properties of the topological relations corresponding to the syntactical expressions.

These eight cases provide some analogues of the JEPD classification as the well-known $RCC8$ one [RCC92]. The following is a representation of them that does not require any restriction on the models.

Thm. 8 Given a modal L-model $\mathcal{M}$, for any negative, non-empty $\alpha, \beta \in \mathit{Wff}$, the following are JEPD:

$\vDash_M \alpha \leftrightarrow \beta$
$\vDash_M \mathsf{ov}'(\alpha, \beta) \wedge \mathsf{ov}'(\alpha, \sim\beta)$
$\vDash_M \mathsf{ntp}(\alpha, \beta)$
$\vDash_M \mathsf{ntp}(\beta, \alpha)$
$\vDash_M \mathsf{tp}(\alpha, \beta) \wedge \mathsf{ov}'(\sim\alpha, \beta)$
$\vDash_M \mathsf{tp}(\beta, \alpha) \wedge \mathsf{ov}'(\alpha, \sim\beta)$
$\vDash_M \mathsf{ntp}(\alpha, \sim\beta)$
$\vDash_M \mathsf{tp}(\alpha, \sim\beta)$

Proof. Similar to lemma 7.

We also get the following, where none of the expressions used to encode the $RCC8$ relations is modal (the only modal expression is τ, used as a postulate for $\mathcal{R}$-nontriviality).

Thm. 9 Given a modal L-model $\mathcal{M}$ s.t. $\vDash_M \sigma$, $\vDash_M \tau$, $\vDash_M \theta$, $\vDash_M \pi$, for any negative, non-empty $\alpha, \beta \in \mathit{Wff}$, the following are JEPD:

$\vDash_M \alpha \leftrightarrow \beta$
$\vDash_M \mathsf{ov}(\alpha, \beta) \wedge \mathsf{ov}(\alpha, \sim\beta)$
$\vDash_M \mathsf{ntp}(\alpha, \beta)$
$\vDash_M \mathsf{ntp}(\beta, \alpha)$
$\vDash_M \mathsf{tp}_r(\alpha, \beta) \wedge \mathsf{ov}(\sim\alpha, \beta)$
$\vDash_M \mathsf{tp}_r(\beta, \alpha) \wedge \mathsf{ov}(\alpha, \sim\beta)$
$\vDash_M \mathsf{ntp}(\alpha, \sim\beta)$
$\vDash_M \mathsf{tp}_r(\alpha, \sim\beta)$

Proof. Similar to lemma 7, using also lemmas 22,25.

8 Conclusion and further work

We have shown how some extensions of intuitionistic propositional logic can be used to encode spatial notions; relevant comparisons can be made with systems in [RCC92] and in [PS98].

The introduction of r-spaces, as topological structures where the dependance on language and partial knowledge can be made explicit, and the general features of intuitionistic models depending on their ordered character, seem to fit in promisingly with aspects related to granularity and uncertainty in spatial representation (as presented, for example, in [SW97]).

From the point of view of expressiveness, the logical representations that we have presented here differ from those in [Ben96, Ben98] (based on intuitionistic propositional logic and modal *S4*) especially insofar jere we can get an object-language encoding of connectedness; using an approach based on topological semantics, this would seem hard to obtain without quantification; on the other hand, it is not the case that quantification is needed for connectedness under an altogether different approach, such as that presented in [Lem96].

From the computational point of view, a major problem in our case is given by the undecidability of intuitionistic 2nd-order propositional logic (see [Gab74] for *C2h*), in contrast with the PSPACE-completeness of its quantifier-free fragment [Sta79]. Anyway, work such as [Pit92] shows that the complexity of 2nd-order reasoning can be reduced in significant cases. Further work should focus on the possibility to extract, from the present encodings, logical representations for automated theorem-proving.

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