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A Categorical Axiomatisation of Region-Based Geometry[*]

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Abstract. *Region Based Geometry* (**RBG**) is an axiomatic theory of qualitative configurations of spatial regions. It is based on Tarski's *Geometry of Solids*, in which the *parthood* relation and the concept of *sphere* are taken as primitive. Whereas in Tarski's theory the combination of mereological and geometrical axioms involves set theory, in **RBG** the interface is achieved by purely 1st-order axioms. This means that the elementary sublanguage of **RBG** is extremely expressive, supporting inferences involving both mereological and geometrical concepts. Categoricity of the **RBG** axioms is proved: all models are isomorphic to a standard interpretation in terms of Cartesian spaces over $\mathbb{R}$.

1. Introduction

Many researchers in the field of Qualitative Spatial Reasoning (QSR) have argued that it is useful to have representations in which *spatial regions* are the basic entities [10, 8]. This ontology contrasts with the approach of classical geometry, where lines, surfaces and regions are typically thought of as sets of points, so that reasoning about them requires the use of set theory. To meet this need several region-based theories have been proposed [18, 1, 17]. However, these theories have been limited to describing topological properties, so the expressive power is much more restricted than point-based geometry.

By adding a *sphere* primitive to Leśneiwski's *Mereology*, Tarski [21] showed how to give a categorical axiomatisation of the geometry of regions whose models are isomorphic to the structure of regular open sets of points in Euclidean point-based geometry. He called this theory the *Geometry of Solids*.[1] In Tarski's system, the interface between point Geometry and the regions of mereology is achieved by identifying points with equivalence classes of concentric spheres (i.e. sets of spheres sharing the same

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[1]This is really a theory of 'regions' or 'volumes', since the entities of the theory may inter-penetrate and the property of solidity is not considered.

centre point). Thus the usual approach, where regions are taken as sets of points, is reversed. While the resulting system is technically interesting, it is at odds with the desire to eliminate (or at least minimise) the need for set theory.

The current paper presents a theory of *Region-Based Geometry* (**RBG**), in which geometry and mereology are combined in a much simpler way. Standard axioms for mereology and geometry are used, apart from a relatively straightforward modification of the geometrical axioms to accommodate the more densely populated domain of discourse of a region-based theory. Once certain definitions have been specified, the interaction between parthood and the geometrical concepts is expressed by a small number of purely 1st-order axioms.

To achieve a categorical theory it is of course necessary to retain the two 2nd-order axioms already present in geometry and mereology (the geometrical continuity axiom and the mereological axiom stating that all non-empty classes of regions have a unique sum). However, the 1st-order sublanguage of **RBG** is extremely expressive, with elementary reasoning spanning both mereological and geometrical concepts.

The primary motivation for the development of **RBG** was to provide a secure ontological foundation (as advocated e.g. in [14]) for theories of spatial information. It may also be of use as a framework within which more computationally oriented representations (e.g. [2, 19, 9, 23]) can be embedded. Since the theory has a categorical interpretation in terms of Cartesian fields over $\mathbb{R}$, it is readily compatible with more traditional representations that employ this classical model of space.

The formulation of **RBG** was influenced by [6], which, drawing on [21], constructs a 1st-order theory of spatial regions based on the primitive predicate *simple region* (satisfied by regions that are homeomorphic to a ball) and relations *congruence* and *strong connection* (two 3-dimensional regions are strongly connected if they share a 2-dimensional surface). A precursor of the current **RBG** theory was given in [9]. Subsequent developments appeared in [4] and [5]. The aim of the current paper is to give a concise but definitive version of the theory together with a detailed proof of its categoricity. Further details of what can be defined within the theory can be found in [3].

2. Mereology

We begin by presenting a formal theory of the parthood relation, $\mathsf{P}(x, y)$. As a basis for the axiomatisation we take the classical Mereology of Leśneiwski [15] (see also [21, 25, 20]):

D1) $\mathsf{DR}(x, y) \equiv_{def} \neg\exists z[\mathsf{P}(z, x) \wedge \mathsf{P}(z, y)]$

D2) $\mathsf{SUM}(X, x) \equiv_{def} \forall y[X(y) \to \mathsf{P}(y, x)] \wedge \neg\exists z[\mathsf{P}(z, x) \wedge \forall y[X(y) \to \mathsf{DR}(y, z)]]$

In **D2**, X is a 2nd-order variable, which can denote any subset of the domain of regions. $X(x)$ is of course true just in case the denotation of x is a member of the set denoted by X.

In addition to the usual principles of classical logic,[2] the system is required to satisfy two specifically mereological postulates:

A1) $\forall x \forall y \forall z[\mathsf{P}(x, y) \wedge \mathsf{P}(y, z) \to \mathsf{P}(x, z)]$

A2) $\forall X[\exists x[X(x)] \to \exists! x[\mathsf{SUM}(X, x)]]$

[2]Since the theory includes 2nd-order monadic variables, one may wish to supplement the formalism some set theory or lambda calculus in order to characterise certain inferences involving 2nd-order reasoning (see [3] for further discussion of this issue).

These ensure firstly that the part relation is transitive and secondly (and slightly controversially) that for any non-empty set of individuals there is a unique individual which is the sum of that set.

Many other useful concepts can be defined in terms of P. For instance we define the universal region $\mathcal{U}$, a binary sum function $+$, proper part (PP), overlap (O) and partial overlap (PO) relations, and relational counterparts of Boolean product, difference and complement functions (these are defined as relations because, since the domain of Mereology does not include an 'empty' region, it is not possible to define them as total functions).

D3) $\forall x[x = \mathcal{U} \leftrightarrow \forall y[\mathsf{P}(y,x)]]$
D4) $\forall x[x = (y+z) \leftrightarrow (\mathsf{P}(y,x) \wedge \mathsf{P}(z,x) \wedge \neg\exists w[\mathsf{P}(w,x) \wedge \mathsf{DR}(w,y) \wedge \mathsf{DR}(w,z)])$
D5) $\mathsf{PP}(x,y) \ \equiv_{def} \ (\mathsf{P}(x,y) \wedge \neg(x=y))$
D6) $\mathsf{O}(x,y) \ \equiv_{def} \ \neg\mathsf{DR}(x,y)$
D7) $\mathsf{PO}(x,y) \ \equiv_{def} \ \mathsf{O}(x,y) \wedge \neg\mathsf{P}(x,y) \wedge \neg\mathsf{P}(y,x)$
D8) $\mathsf{Prod}(x,y,z) \ \equiv_{def} \ \forall w[(\mathsf{P}(w,x) \wedge \mathsf{P}(w,y)) \leftrightarrow \mathsf{P}(w,z)]$
D9) $\mathsf{Diff}(x,y,z) \ \equiv_{def} \ \forall w[(\mathsf{P}(w,x) \wedge \mathsf{DR}(w,y)) \leftrightarrow \mathsf{P}(w,z)]$
D10) $\mathsf{Compl}(x,y) \ \equiv_{def} \ \forall w[\mathsf{P}(w,x) \leftrightarrow \mathsf{DR}(w,y)]$

3. Region-Based Geometry

The theory of *Region-Based Geometry* is directly inspired by Tarski's Geometry of Solids [21]. Following Tarski, we build on Leśneiwski's *mereology* by introducing a new primitive *sphere* predicate, which we write $\mathsf{S}(x)$. In terms of P and S a series of geometrical relationships and concepts are defined and a set of postulates is given. Here and in the rest of the paper we shall often want to quantify over just the spherical regions in the domain. For convenience we introduce the notations:

D11) $\forall^\circ x[\phi] \ \equiv_{def} \ \forall x[\mathsf{S}(x) \rightarrow \phi]$
D12) $\exists^\circ x[\phi] \ \equiv_{def} \ \exists x[\mathsf{S}(x) \wedge \phi]$

The relations of *external tangency* (ET), *internal tangency* (IT), *external diametricity* (ED), *internal diametricity* (ID) and *concentricity* ($x \circledcirc y$) are defined as in [21]. See Fig. 1 for 2D illustrations.

D13) $\mathsf{ET}(a,b) \ \equiv_{def} \ (\mathsf{S}(a) \wedge \mathsf{S}(b) \wedge \mathsf{DR}(a,b) \wedge$
$\forall^\circ xy[(\mathsf{P}(a,x) \wedge \mathsf{P}(a,y) \wedge \mathsf{DR}(b,x) \wedge \mathsf{DR}(b,y)) \rightarrow (\mathsf{P}(x,y) \vee \mathsf{P}(y,x))])$
D14) $\mathsf{IT}(a,b) \ \equiv_{def} \ (\mathsf{S}(a) \wedge \mathsf{S}(b) \wedge \mathsf{PP}(a,b) \wedge$
$\forall^\circ xy[(\mathsf{P}(a,x) \wedge \mathsf{P}(a,y) \wedge \mathsf{P}(x,b) \wedge \mathsf{P}(y,b)) \rightarrow (\mathsf{P}(x,y) \vee \mathsf{P}(y,x))])$
D15) $\mathsf{ED}(a,b,c) \ \equiv_{def} \ (\mathsf{ET}(a,c) \wedge \mathsf{ET}(b,c) \wedge$
$\forall^\circ xy[(\mathsf{DR}(x,c) \wedge \mathsf{DR}(y,c) \wedge \mathsf{P}(a,x) \wedge \mathsf{P}(b,y)) \rightarrow \mathsf{DR}(x,y)])$
D16) $\mathsf{ID}(a,b,c) \ \equiv_{def} \ (\mathsf{IT}(a,c) \wedge \mathsf{IT}(b,c) \wedge$
$\forall^\circ xy[(\mathsf{DR}(x,c) \wedge \mathsf{DR}(y,c) \wedge \mathsf{ET}(a,x) \wedge \mathsf{ET}(b,y)) \rightarrow \mathsf{DR}(x,y)])$
D17) $a \circledcirc b \ \equiv_{def} \ \mathsf{S}(a) \wedge \mathsf{S}(b) \wedge [\ (a=b)$
$\vee\ (\mathsf{PP}(a,b) \wedge \forall^\circ xy[(\mathsf{ED}(x,y,a) \wedge \mathsf{IT}(x,b) \wedge \mathsf{IT}(y,b)) \rightarrow \mathsf{ID}(x,y,b)])$
$\vee\ (\mathsf{PP}(b,a) \wedge \forall^\circ xy[(\mathsf{ED}(x,y,b) \wedge \mathsf{IT}(x,a) \wedge \mathsf{IT}(y,a)) \rightarrow \mathsf{ID}(x,y,a)])\ ]$

Note that each of these relations false if any of their arguments is not a sphere.

We now define some fundamental relations involving spheres:

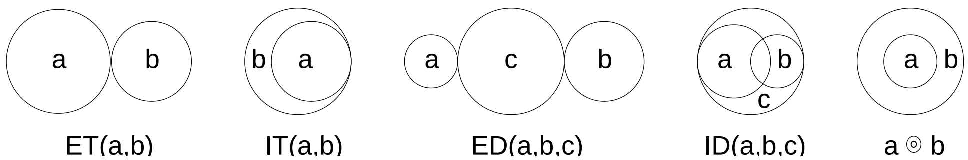

Figure 1. Relations among spheres defined by Tarski

D18) $\mathsf{B}(x,y,z) \ \equiv_{def}\ x \circledcirc y \vee y \circledcirc z \vee$
$\exists x'y'z'vw[x' \circledcirc x \wedge y' \circledcirc y \wedge z' \circledcirc z \wedge \mathsf{ED}(x',y',v) \wedge \mathsf{ED}(v,w,y') \wedge \mathsf{ED}(y',z',w)]$
D19) $\mathsf{COB}(s,r) \ \equiv_{def}\ \mathsf{S}(s) \wedge \forall s'[s' \circledcirc s \rightarrow (\mathsf{O}(s',r) \wedge \neg \mathsf{P}(s',r))]$
D20) $\mathsf{COI}(s,r) \ \equiv_{def}\ \exists s'[s' \circledcirc s \wedge \mathsf{P}(s',r)]$
D21) $\mathsf{EQD}(x,y,z) \ \equiv_{def}\ \exists^\circ z'[z' \circledcirc z \wedge \mathsf{COB}(x,z') \wedge \mathsf{COB}(y,z')]$
D22) $\mathsf{Mid}(x,y,z) \ \equiv_{def}\ (x \circledcirc y \wedge y \circledcirc z) \vee (\mathsf{B}(x,y,z) \wedge \exists^\circ y'[y' \circledcirc y \wedge \mathsf{COB}(x,y') \wedge \mathsf{COB}(z,y')])$
D23) $\mathsf{EQD}(w,x,y,z) \ \equiv_{def}\ \exists^\circ uv[\mathsf{Mid}(w,u,y) \wedge \mathsf{Mid}(x,u,v) \wedge \mathsf{EQD}(v,z,y)]$
D24) $\mathsf{Nearer}(w,x,y,z) \ \equiv_{def}\ \exists^\circ x'[\mathsf{B}(w,x,x') \wedge \neg(x \circledcirc x') \wedge \mathsf{EQD}(w,x',y,z)]$

$\mathsf{B}(x,y,z)$ holds when the centre of y is between the centres of x and z (or coincides with one of these). $\mathsf{COB}(s,r)$ means that sphere s is Centred On the Boundary of r, while $\mathsf{COI}(s,r)$ means that s is Centred on the Interior of r. $\mathsf{EQD}(x,y,z)$ says that the centres of x and y are equidistant from the centre of z. $\mathsf{Mid}(x,y,z)$ says that the centre of y lies mid-way between the centres of x and z; and $\mathsf{EQD}(w,x,y,z)$ holds when the distance between the centres of w and x is the same as the distance between the centres of y and z. $\mathsf{Nearer}(w,x,y,z)$ means that the centres of w and x are closer than the centres of y and z.

Since the concept of the equidistance of two pairs of points (EQD) is definable, we can write the axioms of n-dimensional geometry within our language (the value of n is fixed by appropriate choice of upper and lower dimension axioms). [21] takes this approach to prove that his geometry of solids is categorical and is modelled by n-dimensional Euclidean space in which spheres are interpreted as open balls and 'solids' are regular open sets. We take a similar approach; however, whereas Tarski introduced *points* as sets of spheres, our relations concern spheres but they hold just in case the centre points of the spheres satisfy the corresponding point relations. Thus the quantifiers of the point-based geometry axioms can be replaced by quantifiers over spheres and the equality relation replaced by the $\circledcirc$ relation.

We formulate our geometrical axioms by modifying the theory of *Elementary Geometry* given in [22]. This is well-suited to our purpose because it assumes only points as primitive entities (rather than, say, points and lines, which would further complicate the interface with Mereology). Hence our geometrical axioms are as follows:

A3) $\forall^\circ xy[\mathsf{B}(x,y,x) \rightarrow x \circledcirc y]$
A4) $\forall^\circ xyzu[(\mathsf{B}(x,y,u) \wedge \mathsf{B}(y,z,u)) \rightarrow \mathsf{B}(x,y,z)]$
A5) $\forall^\circ xyzu[(\mathsf{B}(x,y,z) \wedge \mathsf{B}(x,y,u) \wedge \neg(x \circledcirc y)) \rightarrow (\mathsf{B}(x,z,u) \vee \mathsf{B}(x,u,z))]$
A6) $\forall^\circ xy[\mathsf{EQD}(x,y,y,x)]$
A7) $\forall^\circ xyz[\mathsf{EQD}(x,y,z,z) \rightarrow x \circledcirc y]$
A8) $\forall^\circ xyzuvw[(\mathsf{EQD}(x,y,z,u) \wedge \mathsf{EQD}(x,y,v,w)) \rightarrow \mathsf{EQD}(z,u,v,w)]$
A9) $\forall^\circ txyzu\exists^\circ v[(\mathsf{B}(x,t,u) \wedge \mathsf{B}(y,u,z)) \rightarrow (\mathsf{B}(x,v,y) \wedge \mathsf{B}(z,t,v))]$

A10) $\forall^{\circ} txyzu \exists^{\circ} vw[(\mathsf{B}(x,u,,t) \wedge \mathsf{B}(y,u,z) \wedge \neg(x \circledcirc y)) \rightarrow (\mathsf{B}(x,z,v) \wedge \mathsf{B}(x,y,w) \wedge \mathsf{B}(v,t,w))]$

A11) $\forall^{\circ} xx'yy'zz'uu'[(\mathsf{EQD}(x,y,x',y') \wedge \mathsf{EQD}(y,z,y',z') \wedge \mathsf{EQD}(x,u,x',u') \wedge \mathsf{EQD}(y,u,y',u') \wedge \mathsf{B}(x,y,z) \wedge \mathsf{B}(x',y',z') \wedge \neg(x \circledcirc y)) \rightarrow \mathsf{EQD}(z,u,z',u')]$

A12) $\forall^{\circ} xyuv \exists^{\circ} z[\mathsf{B}(x,y,z) \wedge \mathsf{EQD}(y,z,u,v)]$

A13) $\forall XY[\exists^{\circ} z \forall^{\circ} xy[(X(x) \wedge Y(y)) \rightarrow \mathsf{B}(z,x,y)] \rightarrow \exists^{\circ} z \forall^{\circ} xy[(X(x) \wedge Y(y)) \rightarrow \mathsf{B}(x,z,y)]]$

A14) $\forall^{\circ} xyz[(x \circledcirc y \wedge y \circledcirc z) \rightarrow x \circledcirc z]$

A15) $\forall^{\circ} xx'yzw[(\mathsf{EQD}(x,y,z,w) \wedge x' \circledcirc x) \rightarrow \mathsf{EQD}(x',y,z,w)]$

A16^n) $\exists^{\circ}\, x_0 \ldots x_n[\bigwedge_{0 \leq i \neq j \neq k \leq n}(\neg(x_i \circledcirc x_j) \wedge \mathsf{EQD}(x_i,x_j,x_j,x_k))] \wedge \neg\exists^{\circ}\, x_0 \ldots x_{n+1}[\bigwedge_{0 \leq i \neq j \neq k \leq n+1}(\neg(x_i \circledcirc x_j) \wedge \mathsf{EQD}(x_i,x_j,x_j,x_k))]$

A3-12 correspond directly to Tarski's first ten axioms. As explained above quantification is restricted to spheres and equality is replaced by concentricity. Rather than the 1st-order schema version of the continuity axiom, used in [22], we employ a 2nd-order continuity axiom, **A13**. This is required to ensure categoricity, although the weaker version might be more appropriate for reasoning applications.

Axioms **A14** and **A15** ensure that $\circledcirc$ behaves like equality relative to the geometrical axioms (reflexivity and symmetry are implicit in the definition of $\circledcirc$). Reflexivity and symmetry are implicit in the definition of $\circledcirc$, so to ensure it is an equivalence relation only a transitivity axiom needs to be added (it is possible that this too is derivable from the other axioms). Since the meanings of the basic geometrical relations depend only on the centre points of the spheres involved, their truth must be preserved by substituting one variable for another that denotes a concentric sphere. Since all geometrical concepts are definable in terms of equidistance and because of the symmetry properties of EQD, it is sufficient to axiomatise this substitution property only for the case of the first argument of the $\mathsf{EQD}(x,y,z,w)$ relation.

In **RBG** Tarski's upper and lower dimension axioms (which in his system fix the dimension to 2) are replaced by an instance of the schematic formula **A16^n**, where n is the required dimensionality. This formula asserts that in an n dimensional space there are $n+1$ (and no more) mutually equidistant equidistant points. The theory consisting of the geometrical axioms **A3-A15** and **A16^n** (for some natural number n), will be called **G^n**.

To get a categorical axiomatisation one must ensure that the class of regular open sets of centre points of spheres coincides with the class of regions and that the P relation corresponds to the inclusion relation among the centre points. Rather than stating these as meta-level constraints (as Tarski does) we enforce them directly by 1st-order axioms:

A17) $\forall^{\circ} xy[\neg(x \circledcirc y) \rightarrow \exists^{\circ} s \forall^{\circ} z[\mathsf{COI}(z,s) \leftrightarrow \mathsf{Nearer}(x,z,x,y)]$

A18) $\forall^{\circ} x \exists^{\circ} y[\neg(x \circledcirc y) \wedge \forall^{\circ} z[\mathsf{COI}(z,x) \leftrightarrow \mathsf{Nearer}(x,z,x,y)]]$

A19) $\forall xy[\mathsf{P}(x,y) \leftrightarrow \forall s[\mathsf{COI}(s,x) \rightarrow \mathsf{COI}(s,y)]]$

A20) $\forall r \exists^{\circ} s[\mathsf{P}(s,r)]$

The defined relation $\mathsf{COI}(s,r)$ (sphere s is centred at an interior point of region r) provides a means of specifying that each equivalence class of concentric spheres contains all and only those spheres which are centred at some point in space (i.e., as we shall see, in $\mathbb{R}^n$). **A17** ensures that for every two non-concentric spheres x and y there is a sphere concentric with x and bounded by the centre point of y. **A18** says that all spheres can be constructed in this way.

A19 means that $\mathsf{P}(x,y)$ holds just in case every interior point of x is an interior point of y (this actually makes **A1** redundant). This axiom looks like it could be taken as a definition of P but in fact, since COI is itself defined in terms of P and $\circledcirc$, it is really an axiom constraining P relative to the geometrical concepts. On the other hand, one could just as well take COI as a primitive, in which case **A19** would be construed as a definition and **D20** as an axiom.

A20 states that every region has a spherical part (from this it can be proved that every region is equal to the sum of its spherical parts). The theory specified by the axioms **A1–20** we call $\mathbf{RBG}^n$ (n-dimensional Region-Based Geometry).

4. Categorical Interpretation of RBG

In this section it is proved that all models of the theories $\mathbf{RBG}^n$ are isomorphic to a standard interpretation over the Cartesian space $\mathbb{R}^n$. In order to show this a number of other kinds of model structure will be defined.

4.1. Geometrical Models

- A (proper) *n-dimensional geometrical model* is a structure $\mathfrak{G}^n = \langle P, \circledcirc, \mathsf{B}, \mathsf{EQD}\rangle$, where P is a set of elements and $\circledcirc$, B and EQD are respectively binary ternary and quaternary relations over P satisfying the geometrical axioms $\mathbf{G}^n$ (for some dimension n) and also:

  SD) $\forall x[\mathsf{S}(x)]$

  CI) $\forall xy[x \circledcirc y \leftrightarrow x = y]$

  Axiom **S** simply restricts the domain to the class of 'spheres' (so the predicate S is trivially definable and not included in the signature of the structure). **P** makes $\circledcirc$ coincide with the identity relation (so it too is definable and need not be explicitly mentioned in the signature; however, it is included to facilitate the analysis of the more general models defined below).

Lemma 1. Every n-dimensional geometrical structure is isomorphic to the structure $\langle \mathbb{R}^n, \circledcirc, \mathsf{B}, \mathsf{EQD}\rangle$, where each element is identified with a (coordinate) tuple in $\mathbb{R}^n$, $\circledcirc$ is the identity relation and betweenness and equidistance have their usual algebraic definitions in terms of the coordinate tuples.

Proof: It is easy to show that for each n the axioms $\mathbf{G}^n$, **SD**, **CI** are logically equivalent to an n-dimensional version of Tarski's axioms given in [22],[3] which are known to provide a categorical axiom system for the geometry of points, such that all models are isomorphic to the standard interpretation over $\mathbb{R}^n$. □

4.2. Relaxed Geometrical Models

We now define models in which multiple objects may be associated with each geometrical point:

[3] In [22] Tarski is primarily concerned with *elementary* geometries, where the 2nd-order continuity axiom is replaced with a 1st-order schema. He proves that the models of his axioms (with appropriate upper and lower dimension axioms to set fix the dimension to n) are exactly the Cartesian spaces $\mathfrak{F}^n$, where $\mathfrak{F}$ is some *real closed (ordered) field* (betweenness and equidistance having their usual definitions). Since $\mathbb{R}$ is the unique ordered field that is continuous, it is clear that the systems obtained by using the 2nd-order continuity axiom are categorical, with all models being isomorphic to the standard interpretation over $\mathbb{R}^n$.

- A *relaxed n-dimensional geometrical model* is a structure $\mathfrak{H}^n = \langle Q, \circledcirc, \mathsf{B}, \mathsf{EQD}\rangle$, which satisfies $\mathbf{G}^n$ and **SD** but, in place of **CI**, $\circledcirc$ is only required to be an equivalence relation.
- A formula containing only relation symbols in $\{\circledcirc, \mathsf{B}, \mathsf{EQD}\}$ and *not* containing any equality relation will be called a *relaxed geometrical formula*.
- If a pair $\langle q_i, q_j\rangle \in Q^2$ satisfies $x \circledcirc y$ in $\mathfrak{H}^n$ we say that q_i and q_j are *concentric* in $\mathfrak{H}^n$.
- Any substructure of $\mathfrak{H}^n$ whose domain includes exactly one element from each $\circledcirc$ equivalence class in Q will be called a *minimal geometrical substructure*.
- Given any relaxed geometrical model $\mathfrak{H}^n = \langle Q, \circledcirc, \mathsf{B}, \mathsf{EQD}\rangle$ and one of its minimal geometrical substructures $\mathfrak{H}^{n\prime} = \langle Q', \circledcirc', \mathsf{B}', \mathsf{EQD}'\rangle$, the mapping $\mu : Q \to Q'$, such that q and $\mu(q)$ are concentric in $\mathfrak{H}^n$ for every $q \in Q$, is called the *minimisation function* from $\mathfrak{H}^n$ to $\mathfrak{H}^{n\prime}$.

Lemma 2. For any relaxed geometrical model $\mathfrak{H}^n = \langle Q, \circledcirc, \mathsf{B}, \mathsf{EQD}\rangle$ and any 1st-order relaxed geometrical formula $\phi(x_1, \dots, x_j)$, whenever $\langle q_1, .., q_i, .., q_j\rangle \in Q^J$ satisfies $\phi(x_1, \dots, x_j)$ in $\mathfrak{H}^n$, and q'_i is concentric with q_j then $\langle q_1, .., q'_i, .., q_j\rangle$ also satisfies $\phi(x_1, \dots, x_j)$ in $\mathfrak{H}^n$.

Proof: Each case where ϕ is an atomic formula can be proved directly from the geometrical axioms (making particular use of **A15**). The generalisation to all 1st-order relaxed geometrical formulae follows by straightforward induction on formula structure. □

Lemma 3. If $\langle q_1, \dots, q_j\rangle$ satisfies a 1st-order relaxed geometrical formula $\phi(x_1, \dots, x_j)$ in a relaxed geometrical model $\mathfrak{H}^n$ and $\mathfrak{H}^{\mathsf{n}\prime}$ is a minimal geometrical substructure of $\mathfrak{H}^n$ then k $\langle \mu(q_1), \dots, \mu(q_n)\rangle$ satisfies $\phi(x_1, \dots, x_j)$ in $\mathfrak{H}^{\mathsf{n}\prime}$, where μ is the minimisation function from $\mathfrak{H}^n$ to $\mathfrak{H}^{\mathsf{n}\prime}$.

Proof: This follows directly from Lemma 2 and the definitions of minimal geometrical substructure and the minimisation function. □

Lemma 3 has the following immediate corollaries:

Corollary 3a: Every 1st-order relaxed geometrical formula satisfiable in a relaxed geometrical model is satisfiable in any minimal geometrical substructure of that model.

Corollary 3b: Any closed 1st-order relaxed geometrical formula which is true in a relaxed geometrical model is also true in any minimal geometrical substructure of that model.

Lemma 4. Any minimal geometrical substructure of a relaxed n-dimensional geometrical model is a proper n-dimensional geometrical model (and hence is isomorphic to $\mathfrak{G}^n$).

Proof: Let $\mathfrak{H}^n = \langle Q, \circledcirc, \mathsf{B}, \mathsf{EQD}\rangle$ be a relaxed geometrical model. Clearly any minimal geometrical substructure will satisfy **S**. We need to show that it satisfies the geometrical axioms $\mathbf{G}^n$. Corollary 3b guarantees that all the 1st-order geometrical axioms, since they are 1st-order relaxed geometrical formulae, will be satisfied by every minimal geometrical substructure.

To prove that the truth of the 2nd-order continuity axiom **A13** is also preserved, we first show that if **A13** is satisfied in $\mathfrak{H}^n$ then the open formula

A13') $$\forall XY[(\neg\exists z[(X(z) \vee Y(z)) \wedge \neg Z(z)] \wedge \exists^\circ z[Z(z) \wedge \forall^\circ xy[(X(x) \wedge Y(y)) \to \mathsf{B}(z,x,y)]]]) \\ \to \exists^\circ z[Z(u) \wedge \forall^\circ xy[(X(x) \wedge Y(y)) \to \mathsf{B}(x,z,y)]]]$$

is satisfied whenever the free 2nd-order variable Z takes the value of some $S \subseteq Q$, which is the domain of a minimal geometrical substructure of $\mathfrak{H}^n$.

A13' is obtained from **A13** as follows. Addition of the antecedent restricting the universally quantified 2nd-order variables is justified by purely logical inference. We can also see that the satisfiability of the existential sub-formulae is not affected by restricting quantification to the domain of a minimal geometrical substructure, since whenever either of the subexpressions $\forall^{\circ}xy[(X(x) \wedge Y(y)) \rightarrow \mathsf{B}(z,x,y)]$ and $\forall^{\circ}xy[(X(x) \wedge Y(y)) \rightarrow \mathsf{B}(x,z,y)]$, with free variables X, Y, z, are satisfied by $\langle S_1, S_2, q\rangle$ they are also satisfied by every tuple $\langle S_1, S_2, q'\rangle$ such that q' is concentric with q.

A13' is like **A13** but with all domains of quantification explicitly restricted to the set denoted by Z. Hence, if $\langle S\rangle$ satisfies **A13'** in $\mathfrak{H}^n$, it must also satisfy **A13'** in the substructure with domain S. Moreover, on this substructure S is identical with universe of discourse, so the quantifier restrictions are redundant and the substructure must also satisfy the original continuity axiom **A13**. □

The relationship between relaxed geometrical models and the standard $\mathbb{R}^n$ interpretation of the geometrical concepts is exhibited by the following class of mappings:

- For any relaxed geometrical model $\mathfrak{H}^n = \langle Q, ⊚, \mathsf{B}, \mathsf{EQD}\rangle$ a surjective function $\pi : Q \rightarrow \mathbb{R}^n$ is called a *Cartesian centre-point interpretation function* or more briefly a *CCPI-function* iff it fulfills the following condition: For any $q_1, \ldots, q_7 \in Q$, $\langle q_1, q_2, q_3\rangle$ satisfies $\mathsf{B}(x,y,z)$ and $\langle q_4, q_5, q_6, q_7\rangle$ satisfies $\mathsf{EQD}(x,y,z,w)$ in $\mathfrak{H}^n$ just in case the tuples $\langle \pi(q_1), \pi(q_2), \pi(q_3)\rangle$ and $\langle \pi(q_4), \pi(q_6), \pi(q_6), \pi(q_7)\rangle$ satisfy the standard definitions of betweenness and equidistance in $\mathbb{R}^n$; and also $\langle q_1, q_2\rangle$ satisfies $x ⊚ y$ in $\mathfrak{H}^n$ iff $\pi(q_1) = \pi(q_2)$.

Lemma 5. On any relaxed geometrical model $\mathfrak{H}^n = \langle Q, ⊚, \mathsf{B}, \mathsf{EQD}\rangle$ there is CCPI-function, $\pi : Q \rightarrow \mathbb{R}^n$.

Proof: Let $\mathfrak{H}^{n\prime} = \langle Q', ⊚', \mathsf{B}', \mathsf{EQD}'\rangle$ be a minimal geometrical substructure of $\mathfrak{H}^n$ and let μ be the minimisation function from $\mathfrak{H}^n$ to $\mathfrak{H}^{n\prime}$. Lemma 3 ensures that, for each basic relation $R \in \{⊚, \mathsf{B}, \mathsf{EQD}\}$, the tuple $\langle q_1, \ldots, q_j\rangle$ satisfies $R(x_1, \ldots, x_j)$ in $\mathfrak{H}^n$ just in case $\langle \mu(q_1), \ldots, \mu(q_j)\rangle$ satisfies $R(x_1, \ldots, x_j)$ in $\mathfrak{H}^{n\prime}$. By Lemma 4, $\mathfrak{H}^{n\prime}$ is isomorphic to $\mathfrak{G}^n$, which by Lemma 1 is isomorphic to the standard $\mathbb{R}^n$ model. Thus there must be a surjective mapping (π) directly from Q to $\mathbb{R}^n$ that fulfills the requirements of a CCPI-function. □

4.3. Sphere Models

A structure whose elements are isomorphic to the set of all open n-balls in $\mathbb{R}^n$ is specified by:

- An *n-dimensional sphere model* is a structure $\mathfrak{S}^n = \langle S, ⊚, \mathsf{B}, \mathsf{EQD}, \mathsf{COI}\rangle$, where $\langle S, ⊚, \mathsf{B}, \mathsf{EQD}\rangle$ is a relaxed geometrical model and COI also satisfies **A17** and **A18** and also

  ID) $\forall x \forall y[\forall z[\mathsf{COI}(z,x) \leftrightarrow \mathsf{COI}(z,y)] \rightarrow x = y]$.

Since a sphere model is also a relaxed geometrical model, it must support a CCPI-function. In terms of this we define the following useful function:

- A *Cartesian open ball interpretation* function or *COBI-function* $\Pi : S \rightarrow \wp(\mathbb{R}^n)$ for a sphere model $\mathfrak{S}^n = \langle S, ⊚, \mathsf{B}, \mathsf{EQD}, \mathsf{COI}\rangle$ is defined by: $\Pi(s) = \{\pi(s_i) \mid \langle s_i, s\rangle$ satisfies $\mathsf{COI}(x,y)$ in $\mathfrak{S}^n\}$, where π is a CCPI-function on $\mathfrak{S}^n$. We say that Π is *supervenient* upon π.
- Let $B_{\mathbb{R}^n}$ denote the set of open n-balls in $\mathbb{R}^n$.
- For any $p_1, p_2 \in \mathbb{R}^n$ such that $p_1 \neq p_2$, let $\beta(p_1, p_2)$ be the (unique) open n-ball in $\mathbb{R}^n$ that is centred on p_1 and has p_2 as a boundary point.

Lemma 6. In any sphere model with a CCPI-function π, the defined relation $\mathsf{Nearer}(x, y, z, w)$ is satisfied by $\langle s_i, s_j, s_k, s_l\rangle$ just in case $\delta(\pi(s_i), \pi(s_j)) < \delta(\pi(s_k), \pi(s_l))$, where $\delta(p_1, p_2)$ is the distance between points p_1 and p_2 in $\mathbb{R}^n$.

Proof: Since sphere models are a special kind of relaxed geometrical model, Lemma 5 tells us there must be a CCPI-function $\pi : S \to \mathbb{R}^n$ and that π for any $R \in \{\circledcirc, \mathsf{B}, \mathsf{EQD}\}$, $\langle s_1, \ldots, s_j\rangle \in S^j$ satisfies $R(x_1, \ldots, x_j)$ in $\mathfrak{S}^n$ just in case $\langle\pi(s_1), \ldots, \pi(s_j)\rangle \in \mathbb{R}^{n\,j}$ satisfies the standard interpretation of R in $\mathbb{R}^n$. Since Nearer is defined from these primitive relation relations, whether a tuple $\langle s_i, s_j, s_k, s_l\rangle$ satisfies $\mathsf{Nearer}(x, y, z, w)$ is also completely determined by the corresponding points $\pi(s_i)$, $\pi(s_j)$, $\pi(s_k)$, $\pi(s_l) \in \mathbb{R}^n$ and must also have its usual interpretation in terms of these points. □

Lemma 7. For any sphere model $\mathfrak{S}^n = \langle S, \circledcirc, \mathsf{B}, \mathsf{EQD}, \mathsf{COI}\rangle$ with a CCPI-function π, $\langle s_1, s_2, s_3\rangle \in S^3$, where s_2 and s_3 are not concentric in $\mathfrak{S}^n$, satisfies $\phi(x, y, z) \equiv \forall^\circ w[\mathsf{COI}(w, x) \leftrightarrow \mathsf{Nearer}(y, w, y, z)]$ in $\mathfrak{S}^n$ just in case $\Pi(s_1) = \beta(\pi(s_2), \pi(s_3))$.

Proof: This is a straightforward consequence of Lemma 6 and the definition of Π. □

Lemma 8. For any sphere model $\mathfrak{S}^n = \langle S, \circledcirc, \mathsf{B}, \mathsf{EQD}, \mathsf{COI}\rangle$, if π is a CCPI-function on $\mathfrak{S}^n$ and Π is a COBI-function supervenient on π then, for all $s \in S$, $\Pi(s)$ is an open n-ball of $\mathbb{R}^n$ and $\pi(s)$ is the centre point of $\Pi(s)$.

Proof: Applying the result of Lemma 7 to **A18** (and taking into account axiom **S**), we see that for every $s \in S$ we have $\Pi(s) = \beta(\pi(s), \pi(t))$ for some $t \in S$. □

Lemma 9. Every COBI-function Π on a sphere model $\mathfrak{S}^n$ is a bijection onto $B_{\mathbb{R}^n}$.

Proof: By Lemma 8 the range of Π is a subset of $B_{\mathbb{R}^n}$

Axiom **A17** ensures that for every two non-concentric elements $s_1, s_2 \in S$ there is some $s_3 \in S$, such that for any $s_4 \in S$ the pair $\langle s_4, s_3\rangle$ satisfies $\mathsf{COI}(x, y)$ in $\mathfrak{S}^n$ just in case $\langle s_1, s_4, s_1, s_2\rangle$ satisfies $\mathsf{Nearer}(x, y, z, w)$. This means that the set $\Pi(s_3)$ is identical with the n-ball $\beta(\pi(s_1), \pi(s_2))$. Furthermore, since π is surjective, for any two distinct points $p_i, p_j \in \mathbb{R}^n$, we can find non-concentric elements $s_i, s_j \in S$ such that $\pi(s_i) = p_i$ and $\pi(s_j) = p_j$; and consequently, for any open n-ball $B \subset \mathbb{R}^n$ there is an element $s_B \in S$ such that $\Pi(s_B) = B$. Thus, Π is surjective onto $B_{\mathbb{R}^n}$.

From **ID** and the definition of Π it immediately follows that if $\Pi(s_1) = \Pi(s_2)$ then $s_1 = s_2$. Thus Π is also one-one and therefore a bijection. □

Lemma 10. For any sphere model $\mathfrak{S}^n = \langle S, \circledcirc, \mathsf{B}, \mathsf{EQD}, \mathsf{COI}\rangle$ any COBI-function $\Pi : S \to \wp(\mathbb{R}^n)$ fulfills the following conditions: $\langle s_i, s_j\rangle$ satisfies $x \circledcirc y$ iff $\Pi(s_i)$ and $\Pi(s_j)$ are concentric in the usual sense; $\langle s_i, s_j, s_k\rangle$ satisfies $\mathsf{B}(x, y, z)$ iff the centre point of $\Pi(s_j)$ is geometrically between the centre points of $\Pi(s_i)$ and $\Pi(s_k)$; $\langle s_i, s_j, s_k, s_l\rangle$ satisfies $\mathsf{EQD}(x, y, z, w)$ iff the distance between the centre points of $\Pi(s_i)$ and $\Pi(s_j)$ is the same as the distance between the centre points of $\Pi(s_k)$ and $\Pi(s_l)$; $\langle s_i, s_j\rangle$ satisfies $\mathsf{COI}(x, y)$ iff the centre point of $\Pi(s_i)$ lies within $\Pi(s_j)$.

Proof: Lemma 7 established that for every $s \in S$ the centre point of $\Pi(s)$ is $\pi(s)$, where π is the CCPI-function upon which Π supervenes. The definition of a CCPI-function then guarantees that, for the geometrical relations $\circledcirc$, B and EQD, Π must satisfy the requirements of the lemma. From the definition of Π, it follows that $\langle s_1, s_2\rangle$ satisfies $\mathsf{COI}(x, y)$ in $\mathfrak{S}^n$ just in case $\pi(s_1) \in \Pi(s_2)$. □

Lemma 11. All n-dimensional sphere models are isomorphic to $\langle B_{\mathbb{R}^n}, ⊚, \mathsf{B}, \mathsf{EQD}, \mathsf{COI}\rangle$, where: $\langle s_i, s_j\rangle \in (B_{\mathbb{R}^n})^2$ satisfies $x⊚y$ iff s_i and s_j are concentric; for any $R \in \{⊚, \mathsf{B}, \mathsf{EQD}\}$, $\langle s_1, \ldots, s_j\rangle \in S^j$ satisfies $R(x_1, \ldots, x_j)$ iff the centre points of $s_1, \ldots, s_j$ satisfy the standard interpretation of R in $\mathbb{R}^n$; $\langle s_i, s_j\rangle$ satisfies $\mathsf{COI}(x, y)$ iff the centre point of s_i lies within the n-ball s_j.

Proof: Lemmas 9 and 10 ensure that COBI-functions (which always exist because of Lemma 5) provide the required isomorphism from any arbitrary sphere model to a sphere model over the domain $B_{\mathbb{R}^n}$.

4.4. A Model for RBG

Finally, models for the whole **RBG** theory are specified by:

- An n-dimensional **RBG**^{n}-model is a structure $\langle R, \mathsf{P}, \mathsf{S}\rangle$, where P and S are respectively binary and unary relations satisfying the axioms of **RBG**^{n}.

To demonstrate categoricity of the axiom system these models are considered as incorporating a sphere model as a substructure. Hence the following definitional expansion of an **RBG**^{n}-model is defined:

- A *verbose* **RBG**^{n}-model is a structure $\mathfrak{M}^n = \langle R, \mathsf{P}, \mathsf{S}, ⊚, \mathsf{B}, \mathsf{EQD}, \mathsf{COI}\rangle$ satisfying all the axioms of **RBG**^{n}.

Lemma 12. Given a verbose **RBG**^{n}-model $\mathfrak{M}^n = \langle R, \mathsf{P}, \mathsf{S}, ⊚, \mathsf{B}, \mathsf{EQD}, \mathsf{COI}\rangle$, the substructure $\langle S, ⊚, \mathsf{B}, \mathsf{EQD}, \mathsf{COI}\rangle$, where $S = \{r \in R \mid \mathsf{S}(r)\}$, is an ($n$-dimensional) sphere model.

Proof: $\mathfrak{M}^n$ satisfies **G**^{n}, **A17**, **A18** and the domains of all quantifiers in these axioms are restricted to elements satisfying $\mathsf{S}(x)$. Thus they must be satisfied on the substructure of elements satisfying $\mathsf{S}(x)$. **ID** is satisfied by $\mathfrak{M}^n$ because it is a theorem of **RBG** (it follows from **A2** and **A19**). Being a purely universal formula it must also be satisfied on all substructures of $\mathfrak{M}^n$. □

The embedded sphere model enables us to generalise the $COBI$-functions of the previous section to functions operating on the larger domain of an **RBG**^{n}-model:

- A *Cartesian regular open set interpretation function* or *CROSI-function* for a **RBG**^{n}-model $\mathfrak{M}^n$ is defined by $\Pi(r) = \{\pi(r_i) \mid \langle r_i, r\rangle$ satisfies $\mathsf{COI}(x, y)$ in $\mathfrak{M}^n\}$, where π is a CCPI-function for the embedded sphere model within $\mathfrak{M}^n$.

Some useful lemmas concerning Π will now be proved:

Lemma 13. Each $r \in R$ satisfies $\mathsf{S}(x)$ in $\mathfrak{M}^n$ if and only if $\Pi(r)$ is an n-ball.

Proof: This follows from Lemmas 9 and 12. □

Lemma 14. A pair $\langle r_1, r_2\rangle \in R^2$ satisfies $\mathsf{P}(x, y)$ in $\mathfrak{M}^n$ just in case $\Pi(r_1) \subseteq \Pi(r_2)$.

Proof: This is an immediate consequence of **A19** and the definition of Π. □

Lemma 15. For any $r_1, r_2 \in R$ such that both r_1 and r_2 satisfy $\mathsf{S}(x)$ in $\mathfrak{M}^n$, the pair r_1, r_2 satisfies $\mathsf{COI}(x, y)$ in $\mathfrak{M}^n$ just in case the centre point of the open n-ball $\Pi(r_1)$ lies within the open n-ball $\Pi(r_2)$.

Proof: Follows from Lemma 13 and the definition of a CROSI-function. □

Lemma 16. For any $r_1, r_2 \in R$, if $\Pi(r_1) \cap \Pi(r_2) = \emptyset$ then $\langle r_1, r_2 \rangle \in R^2$ satisfies $\mathsf{DR}(x, y)$ in $\mathfrak{M}^n$.

Proof: By definition $\mathsf{DR}(x, y) \leftrightarrow \neg\exists z[\mathsf{P}(z, x) \wedge \mathsf{P}(z, y)]$. Using **A20** and the transitivity of P we can derive from this: $\mathsf{DR}(x, y) \leftrightarrow \neg\exists^\circ z[\mathsf{P}(z, x) \wedge \mathsf{P}(z, y)]$. If $\Pi(r_1) \cap \Pi(r_2) = \emptyset$ then there is no n-ball in $\mathbb{R}^n$ which is a subset of both $\Pi(r_1)$ and $\Pi(r_2)$. Hence there can be no element $s \in R$, such that $\langle r_1, r_2, s \rangle$ satisfies $(\mathsf{S}(z) \wedge \mathsf{P}(z, x) \wedge \mathsf{P}(z, y))$. Consequently $\langle r_1, r_2 \rangle$ must satisfy $\neg\exists^\circ z[\mathsf{P}(z, x) \wedge \mathsf{P}(z, y)]$, which by **D1** is equivalent to $\mathsf{DR}(x, y)$. □

Lemma 17. For every non-empty regular open set $O \subseteq \mathbb{R}^n$ there is an element $r \in R$ such that $\Pi(r) = O$.

Proof: Let O be any non-empty regular open set of $\mathbb{R}^n$. Then there is a non-empty set B of open n-balls of $\mathbb{R}^n$ such that $\bigcup B = O$. By Lemma 9, for each $b \in B$ there is some $r \in R$ such that $\Pi(r) = b$. Let $T = \{t_i \in R \mid \Pi(t_i) \in B\}$. T is clearly non-empty. **A2** then guarantees that there is a unique element $t \in R$ such that $\langle T, t \rangle$ satisfies $\mathsf{SUM}(X, x)$. According to the definition of SUM (**D2**), for each $t_i \in T$ the pair $\langle t_i, t \rangle$ satisfies $\mathsf{P}(x, y)$ and hence $\Pi(t_i) = b_i \subseteq \Pi(t)$. Thus $\Pi(t)$ includes all the points of the n-balls in B. So $O \subseteq \Pi(t)$.

We now need to show that $\Pi(t) \subseteq O$. Consider an arbitrary $p \in \Pi(t)$. From the definition of Π there is some $s \in R$ such that s satisfies $\mathsf{S}(x)$, $\langle s, t \rangle$ satisfies $\mathsf{COI}(x, y)$ and $\pi(s) = p$. From **A1**, **A2**, **D17** and **D20** one can prove:

T1) $\forall^\circ s \forall r[\mathsf{COI}(s, r) \leftrightarrow \exists^\circ s'[\mathsf{P}(s', r) \wedge \mathsf{COI}(s, s')]]$,

Thus, $\langle s, t \rangle$ must also satisfy $\phi(x, y)$, where $\phi(x, y) \equiv \exists^\circ s'[\mathsf{P}(s', y) \wedge \mathsf{COI}(x, s')]]$. Applying Lemmas 13, 14 and 15 shows that this formula is satisfied just in case the centre point of $\Pi(s)$ (i.e. p) lies within some open n-ball which is a subset of $\Pi(t)$.

Suppose (with the aim of deriving a contradiction) that p lies outside O. Then there must be some open n-ball that is a subset of $\Pi(t)$ but not of O. Since O is regular, this can only be the case if there is some open n-ball b such that $b \subseteq \Pi(t)$ and b is disjoint from O. But, because of the bijection between sphere structures and n-balls, for every for open n-ball b there is some region region r_b, such that $\Pi(r_b) = b$. And, because $\Pi(r_b) \subseteq \Pi(t)$, the pair $\langle r_b, t \rangle$ must satisfy $\mathsf{P}(x, y)$. On the other hand, since it is disjoint from O, b must be disjoint from all the n-balls $b_i \in B$. Hence by Lemma 16, for each $t_i \in T$ the pair $\langle r_b, t_i \rangle$ satisfies $\mathsf{DR}(x, y)$. However, if $\langle T, t \rangle$ satisfies $\mathsf{SUM}(X, x)$ then according to definition **D2** of SUM it must also satisfy $\phi(X, y)$ where $\phi(X, y) \equiv \neg\exists z[\mathsf{P}(z, x) \wedge \forall y[X(y) \rightarrow \mathsf{DR}(y, z)]]$. But from the facts we have established about r_b it is clear that $\langle T, t, r_b \rangle$ satisfies $\psi(X, x, z)$ where $\psi(X, x, z) \equiv (\mathsf{P}(z, x) \wedge \forall y[X(y) \rightarrow \mathsf{DR}(y, z)])$, thus contradicting the assumption that $\langle T, t \rangle$ satisfies $\mathsf{SUM}(X, x)$.

The argument of the preceding paragraph proves that $\Pi(t) \subseteq O$, so we must in fact have $\Pi(t) = O$, which proves the lemma. □

Lemma 18. For every $r \in R$, $\Pi(r)$ is a non-empty regular open subset of $\mathbb{R}^n$.

Proof: Because of **A20** for any $r \in R$ there is some $s \in R$ such that $\langle s, r \rangle$ satisfies $\phi(x, y) \equiv \mathsf{S}(x) \wedge \mathsf{P}(x, y)$. Lemmas 13 and 14 then ensure that $\Pi(r)$ must be non-empty.

Let $\Pi(r) = S_r$ and let S'_r be the smallest regular open subset such that $S_r \subseteq S'_r$. By Lemma 17 there must be some $r' \in R$ such that $\Pi(r') = S'_r$. And by Lemma 14 $\langle r, r' \rangle$ must satisfy $\mathsf{P}(x, y)$. The nature of regularity and the topology of $\mathbb{R}^n$ mean that there can be no open n-ball which is a subset of S'_r and is also disjoint from S_r. So by appealing to Lemmas 14, 13 and 16 we see that $\langle r, r' \rangle$ must satisfy

$\phi(x,y)$, where $\phi(x,y) \equiv \neg\exists^\circ s[\mathsf{DJ}(s,x) \wedge \mathsf{P}(s,y)]$. However, a well-known theorem of Mereology is $\forall sr[\mathsf{PP}(s,r) \rightarrow \exists x[\mathsf{DJ}(x,s) \wedge \mathsf{P}(x,r)]]$ (the *weak supplementation principle* — see [20]) and from this, using **A20**, one can immediately derive:

T2) $\forall sr[\mathsf{PP}(s,r) \rightarrow \exists^\circ x[\mathsf{DJ}(x,s) \wedge \mathsf{P}(x,r)]]$

Hence, $\langle r, r' \rangle$ cannot satisfy $\mathsf{PP}(x,y)$ so must satisfy $\neg\mathsf{PP}(x,y)$, which is equivalent by definition to $\neg(\mathsf{P}(x,y) \wedge \neg(x = y))$ and thus to $\mathsf{P}(x,y) \rightarrow x = y$. Since the tuple satisfies $\mathsf{P}(x,y)$ it must then satisfy the logical identity $x = y$. Consequently, $r = r'$ and so r is regular open. □

Lemma 19. Π is a bijection from R onto the non-empty regular open subsets of $\mathbb{R}^n$.

Proof: Lemmas 17 and 18 ensure that any CROSI-function Π is surjective onto the non-empty regular open subsets of $\mathbb{R}^n$. Since **ID** is derivable from **RBG**^{n}, Π must also be one-one. □

The main theorem of this paper can now be stated:

Theorem 1. Axioms **RBG**^{n} provide a categorical axiom system for n-dimensional region-based geometry, such that every **RBG**^{n}- model is isomorphic to the structure $\mathfrak{R}^n = \langle R_{\mathbb{R}^n}, \mathsf{P}, \mathsf{S} \rangle$, where: $R_{\mathbb{R}^n}$ is the set of non-empty regular open subsets of $\mathbb{R}^n$; $\langle r_1, r_2 \rangle$ satisfies $\mathsf{P}(x,y)$ in $\mathfrak{R}^n$ iff $r_1 \subseteq r_2$; r satisfies $\mathsf{S}(x)$ in $\mathfrak{R}^n$ iff r is an open n-ball of $\mathbb{R}^n$.

Proof: This follows immediately from Lemmas 13, 14 and 19. □

5. Undecidability of RBG

Examination of the proof theory of **RBG** is beyond the scope of the current paper. However, the following theorem from [5] is worth mentioning:

Theorem 2. For $n \geq 2$, **RBG**^{n} is undecidable.[4]

Proof Sketch: This follows from the observations of [13], who showed how an essentially undecidable set of arithmetical axioms could be encoded in a topological theory whose primitives are easily definable within **RBG**. Undecidability can also be shown *via* the result of [11].

From the result of [13] it also follows that the elementary sub-theory of **RBG** obtained by replacing the 2nd-order axioms by 1st-order axiom schemas is undecidable (and hence also incomplete). Consequently, the proof of some elementary theorems of **RBG** must depend essentially upon 2nd-order reasoning. This situation contrasts with the case of the purely geometrical axioms, whose 1st-order sub-theory is complete and decidable [22].

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[4]Whether **RBG**^{1} is decidable is not currently known to the author; however, this may well be a corollary of some established result concerning reasoning about subsets of $\mathbb{R}$.

6. Conclusion

Towards the end of [21] Tarski suggests that: "The postulate system given above [i.e. for his Geometry of Solids] is far from being simple and elegant; it seems very likely that this postulate system can be essentially simplified." I claim that the theory of Region Based Geometry provides such a simplification.

Aside from its theoretical interest, **RBG** provides a securely founded and very general ontological framework for representing qualitative spatial information. Its expressive capabilities have already been demonstrated in [3] and [4], where the theory is used to define high-level kinematic concepts for describing possible motions of rigid bodies. It also seems to be well-suited to the classification of qualitative shape [7].

The 2nd-order nature of **RBG** poses severe problems for automated reasoning. For many practical applications one would much prefer a tractable or at least decidable formalism. Nevertheless, the theory may be useful as an interlingua within which more computationally effective representations, such as those of [2, 19, 9, 23] can be embedded.

Although **RBG** is extremely general, it does have the limitation that it can only deal with a domain of entities having a given fixed dimension. For many applications it would be useful to have an even more comprehensive theory enabling one to refer to entities of different dimensionality (as in e.g. [12]). Another possible modification of the theory would be to take one of the dimensions as corresponding to the flow of time, in order to formulate a spatio-temporal ontology [16, 24].

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