INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS
Int. J. Numer. Meth. Fluids 2007; **00**:1–6 *Prepared using* fldauth.cls *[Version: 2002/09/18 v1.01]*

An investigation of the film thickness calculation for elastohydrodynamic lubrication problems

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SUMMARY

Two algorithms for solving the elastohydrodynamic lubrication problem are compared in this paper. The first, a so-called *decoupled* method, employs an integral form for the calculation of the elastic deformation of the solid surfaces [1]. The second, a *coupled* method, uses either the integral form or a reformulation of the deflection computation into an equivalent differential form [2]. Results are presented which contrast the accuracy of the two approaches and consider their efficient numerical implementation. Copyright © 2007 John Wiley & Sons, Ltd.

KEY WORDS: EHL; differential deflection; Newton-Krylov

1. INTRODUCTION

Elastohydrodynamic Lubrication, known as EHL, is concerned with the modelling of lubrication problems where the extremely high pressure generated causes the contacting surfaces to deform. In the absence of a lubricant the surfaces will be in rolling contact with each other and the large frictional force will reduce the efficiency of the component and also increase the wear. To reduce this wear an appropriate lubricant is applied between the surfaces. Figure 1 illustrates a typical EHL solution where the lubricant flows from left to right. The pressure becomes large in the contact region but, additionally, a second maximum, the so-called *Petrusevich spike* [1], is visible near the exit of the high-pressure region. Beyond that the pressure falls to zero, the point at which the pressure reaches zero being termed the cavitation point. The film thickness solution shows that the undeformed parabolic profile is flattened in the high-pressure region and a further deformation is produced at a position corresponding to the pressure spike. These deformations are considered to be perfectly elastic and the undeformed shape is recovered as the moving surface exits the contact region.

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Received 26 March 2007
Revised 26 March 2007

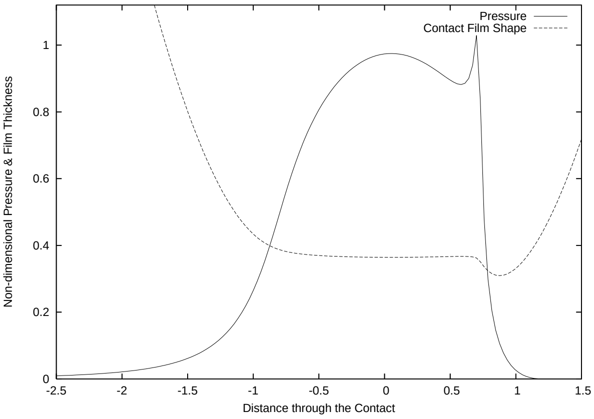


Figure 1. A typical solution for pressure and film thickness in the EHL contact

2. MATHEMATICAL MODEL

The EHL problem is modelled by two main groups of equations: first those concerned with the physical model of the lubricant used; and second those governing the lubrication problem itself. The lubricant model specifies the dependence of the fluid viscosity (η) and density (ρ) on the pressure (P). These are empirical relations and are highly nonlinear since most lubricants are non-Newtonian fluids [1]. The governing equations consist of the following: the Reynolds equation, the Film Thickness equation and the Force Balance equation. The equation system is derived from a non-dimensional, thin-film approximation of Stokes' equation, coupled with a linear elastic model of the contacting surfaces. The inlet boundary and the free cavitation point boundary define the region where the Reynolds Equation is valid. The location of the cavitation point must be determined as part of the solution algorithm [1].

The non-dimensional Reynolds equation reads

$$\frac{d}{dX}\left(\epsilon(P)\frac{dP}{dX}\right) - \frac{d(\rho H)}{dX} \;=\; 0 \;, \tag{1}$$

where $\epsilon = \frac{\rho H^3}{\lambda\eta}$, $P(X)$ and $H(X)$ are the unknown pressure and film thickness and λ is a dimensionless speed parameter. The domain of the problem is from the inlet X_1 to the cavitation point X_c. Boundary conditions of zero pressure are imposed at X_1 and X_c.

The nondimensional Film Thickness equation is given, in integral form [1], as

$$H(X) \;=\; H_{00} + \frac{X^2}{2} - \frac{1}{\pi}\int_{X_1}^{X_c} \ln|X - X'|\, P(X')dX' \;, \tag{2}$$

where H_{00} is the central offset film thickness, the second term defines the undeformed contact shape and the integral term represents the elastic deformation of the contact (also termed the deflection). Evans and Hughes [2] have shown that this equation can, alternatively, be formulated in a differential form as

$$\frac{d^2H(X)}{dX^2} = 1+\frac{1}{\pi}\left(\int_{X_1}^{X-\delta/2}\frac{P(X')}{(X-X')^2}dX' + \int_{X+\delta/2}^{X_c}\frac{P(X')}{(X-X')^2}dX' - 4\frac{P(X)}{\delta} + \frac{\delta}{2}\frac{d^2P(X)}{dX^2}\right) , \tag{3}$$

where the right-hand-side integrals can be more easily approximated than those in (2) and the pressure is approximated locally by a parabola. An alternative approach considered here uses adaptive quadrature for computing the right-hand-side integrals in (3) and makes a locally linear approximation to the pressure.

The nondimensional Force Balance equation, given by

$$\int_{X_1}^{X_c} P(X)dX = \frac{\pi}{2} , \tag{4}$$

represents the balance between the applied load and the total internal pressure in the lubricant.

The nondimensional form for viscosity $\eta(P)$, which was established by Roelands [3], and density $\rho(P)$, which was presented by Dowson and Higginson [4], are

$$\eta(P) = e^{\left(\frac{\alpha p_0}{z}\left[-1+\left(1+\frac{Pp_h}{p_0}\right)^z\right]\right)} , \qquad \rho(P) = \frac{0.59\times 10^9+1.34Pp_h}{0.59\times 10^9+Pp_h} , \tag{5}$$

where z is the viscosity index, α is the pressure viscosity index, p_0 is the ambient pressure and p_h is the maximum Hertzian pressure. The three non-dimensional physical parameters that characterise the line contact problem are velocity (U), load force (W) and elasticity (G).

3. NUMERICAL MODEL

3.1. Discretisation

The spatial domain $x \in [X_1, X_2]$ is discretised with a uniform grid of n points x_i $(1 \le i \le n)$. We consider the cavitation point X_c to be located at an unknown internal point x_j, $2 \le i \le n-1$. We discretise the governing equations using standard techniques. The Reynolds equation (1) is approximated with finite differences and a Trapezoidal integration rule is used for the Force Balance equation (4).

The integral form of the Film Thickness equation (2) can be approximated at point x_i on the regular grid in the following form

$$H(x_i) = H_{00}+\frac{x_i^2}{2}-\frac{1}{\pi}\sum_{j=1}^{n}K_{ij}P(x_j) , \tag{6}$$

where the coefficients K_{ij} can be precomputed approximately through quadrature [1]. Multilevel schemes have been developed to compute the sum in (6) more efficiently [1].

The differential form (3) is approximated on a regular grid in a similar form

$$\frac{d^2H(x_i)}{dX^2} = 1+\frac{1}{\pi}\sum_{j=1}^{n}f_jP(x_j) , \tag{7}$$

and a discrete tridiagonal approximation to the second derivative is solved to recover the film thickness values. The f_j coefficients can be computed either by analytical approximation or by adaptive numerical quadrature.

3.2. Decoupled method

Algorithm 1: The decoupled solution method.

1. Begin with an initial guess for P, H_{00} and the cavitation point position X_c.
2. Evaluate H from the Film Thickness equation (2) or (3).
3. Solve the Reynolds equation (1) for P.
4. Update H_{00} using the Force Balance equation (4).
5. Move the cavitation point based upon the value of $\frac{dP}{dX}$ at the cavitation point.
6. While not converged goto 2.

Multigrid is commonly used to solve the Reynolds equation efficiently [1]. Since EHL problems are nonlinear the Full Approximation Scheme is appropriate [5]. The overall algorithm is such that the cavitation point is only updated on the finest grid and H_{00} is only updated on the coarsest so as to ensure smooth convergence of the whole scheme. Numerical under-relaxation parameters for pressure (C_1) and film thickness (C_2) are applied at steps 3 and 4 respectively.

3.3. Coupled method

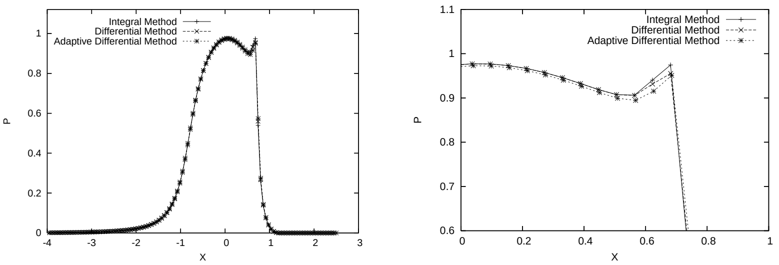
Algorithm 2: The coupled solution method.

1. Begin with an initial guess for the cavitation point position X_c.
2. Solve the coupled equation system (1), (2) or (3) and (4) for P, H and H_{00}.
3. Move the cavitation point based upon the value of $\frac{dP}{dX}$ at the cavitation point.
4. While not converged goto 2.

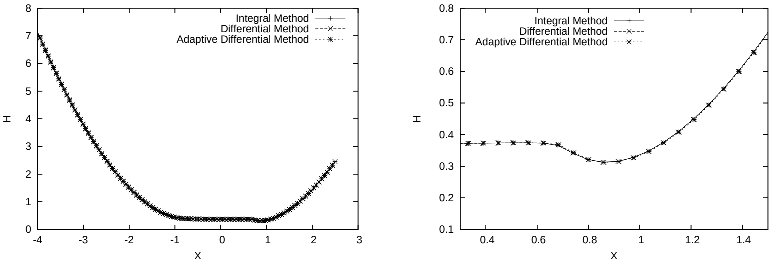
The Kinsol package from Sundials [6] is used to solve the nonlinear equation system at stage 2. Given an initial guess, Kinsol uses an Inexact Newton method which relies on an interior linear equation solver. We can choose a direct solver, which solves the linear system exactly, or an iterative solver, which solves it approximately. In using a direct solver costs grow quickly as the number of points n is increased so, in practice, an iterative solver with appropriate preconditioning is preferred over a direct solver.

4. RESULTS

The EHL system (1)-(4) is solved on the domain $[X_1, X_2] = [-5, 2.5]$, with the following fluid parameters: $\alpha = 2.165 \times 10^{-8}$; $U = 2.0e^{-11}$; $W = 4.0e^{-5}$; $G = 5.0e^{3}$; $C_1 = 0.4$; $C_2 = 0.03$; $z = 0.6$; $p_0 = 1.98 \times 10^{8}$; and $p_h = 5.8 \times 10^{8}$. Typical results, with $n = 129$ points, are shown in Figure 2. These results show that the extra cost associated with the adaptive quadrature approximation do not yield any significant improvements in the evaluation of the deflection. Figure 3 illustrates a close up of the pressure solution on three different grids in the vicinity of the pressure spike. We can see that as the grids are refined, the solutions become closer to each other, and the different approaches for computing the deflection all give similar results.



(a) Pressure solution (b) Detail around the pressure spike



(c) Film thickness solution (d) Detail around the pressure spike

Figure 2. Solution of the EHL problem using the different algorithms

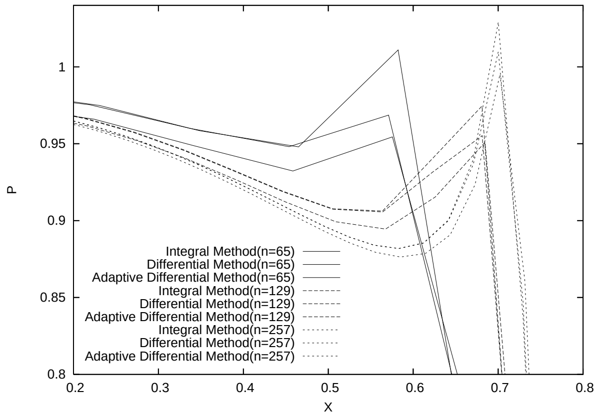


Figure 3. Solution of the EHL problem on three different grid points using the different algorithms

Most practical implementations of the decoupled method make use of both multigrid (for the Reynold's equation) and multi-level multi-integration (for the film thickness equation) in order to produce a robust and efficient solution, [1]. With these multilevel schemes this method is very fast at solving the EHL problem, however it involves careful selection of numerical parameters to obtain solutions in a robust manner. Furthermore, the multi-level multi-integration requires a regular grid. Without the use of these multilevel schemes, not only is the method slower to converge but it also appears to require a fine tuning of the numerical parameters in order to get

n	U	W	G	Coupled converged?	Decoupled converged?
60	$2.0 \times e^{-11}$	$4.0 \times e^{-5}$	$5.0 \times e^{3}$	yes	yes
60	$2.0 \times e^{-10}$	$4.0 \times e^{-5}$	$5.0 \times e^{3}$	yes	no
60	$2.0 \times e^{-11}$	$4.0 \times e^{-4}$	$5.0 \times e^{3}$	yes	no
120	$2.0 \times e^{-11}$	$4.0 \times e^{-5}$	$5.0 \times e^{3}$	yes	yes
120	$2.0 \times e^{-10}$	$4.0 \times e^{-5}$	$5.0 \times e^{3}$	yes	no
120	$2.0 \times e^{-11}$	$4.0 \times e^{-4}$	$5.0 \times e^{3}$	yes	yes

Table I. An illustration of sensitivity of the convergence of two schemes to the physical parameters

convergence when the physical parameters are altered. To illustrate this Table 4 shows sample results of convergence/non-convergence under different physical conditions when the numerical parameters are left unchanged. The coupled approach is found to be more robust than the decoupled approach. One of the advantages of this method is that the film thickness can be calculated with either formulation of the Film Thickness equation, wheras convergence of the decoupled method appears to be even more sensitive to the choice of numerical parameters when the differential method is used. The price that one must pay for the increased robustness of the coupled approach is that it involves larger nonlinear systems.

5. CONCLUSIONS AND FUTURE WORK

In this paper two different algorithms for solving the EHL problem with alternative approaches for computing the film thickness were introduced. We can conclude the following features:

- The calculation of the right-hand side for the deflection method using adaptive quadrature does not yield sufficient improvement to justify the increased cost.
- For a fixed choice of numerical parameters, the coupled method is more robust than the decoupled method.

Current work seeks to exploit the robustness of the coupled method, along with the observation that the dense block in the Jacobian matrix associated with the film thickness calculation is more diagonally dominant for the differential method than for the integral method, to obtain more efficient preconditioning for the Newton-Krylov iterative solve within Kinsol.

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