School of Computing Computational PDEs Unit

DISCONTINUOUS GALERKIN METHODS and IMPLICIT-EXPLICIT TIMESTEPPING

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- Introduction to DG METHODS
- LINEAR ADVECTION
- ADVECTION REACTION PROBLEMS
- CURRENT RESEARCH IN DG METHODS

Acknowlegement Much of this talk is based on the paper *Implicit-Explicit Time Stepping with Spatial Discontinuous Finite Elements* by Willem Hundsdorfer (CWI Amsterdam) and Jérôme Jaffré (INRIA-Rocquencourt) Applied Numerical Mathematics, 2003.

LINEAR DG APPROXIMATION METHODS

Consider $u_t + f(u)_x = g(u)$ with $0 \le x \le L$, $0 \le t \le T$ and initial/boundary data.

Discretise in space giving $w(t) = u_h(t)$ and time step to get $w_n = u_{h,n}$ at time t_n.

Mesh cells $\mathscr{C}_i = (x_{i-1/2}, x_{i+1/2})$ with midpts x_i and $h_i = x_{i+1/2} - x_{i-1/2}$.

$w = u_h \in M^1$, space of piecewise linear discontinuous fns.

$$\int_{\mathscr{C}_i} w_t(x,t)\, v(x)\, dx - \int_{\mathscr{C}_i} f(w(x,t))\, v_x(x)\, dx - f^*_{i-1/2}(t)\, v(x^+_{i-1/2})$$

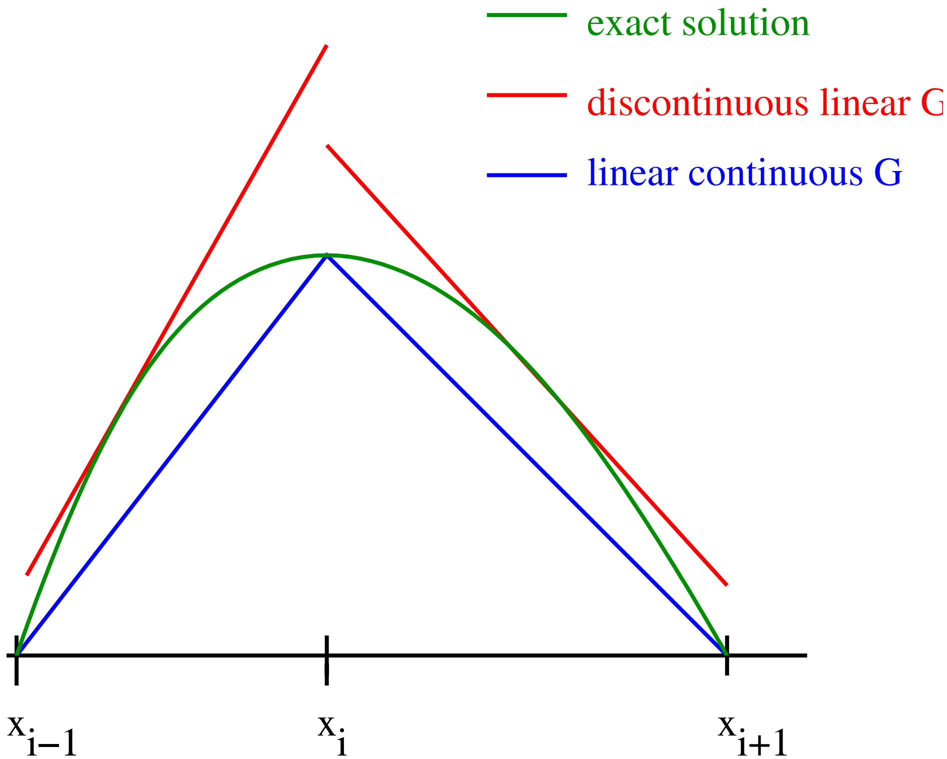
$$+\ f^*_{i+1/2}(t)\, v(x^-_{i+1/2})\ =\ \int_{\mathscr{C}_i} g(w(x,t))\, v(x)\, dx\ \forall\ v \in M^1.$$

Fluxes $f^*_{i+1/2}(t)$ constructed by flux splitting using left and right values $w(x^-_{i+1/2}, t)$ and $w(x^+_{i+1/2}, t)$, as in Finite Volume schemes (e.g. Godunov or Engquist Osher)

Scalar eqns with increasing f - upstream fluxes $f^*_{i+1/2}(t) = f(w(x^-_{i+1/2}, t))$.

BASIS: Let $M^1 = M^0 + \widetilde{M}^1$ with $w(x,t) = \overline{w}_i(t) + \widetilde{w}_i(t)\varphi_i(x)$ with $\varphi_i(x) = 2h_i^{-1}(x - x_i)$.

Allowing discontinuities is important for approximation of steep gradients



Only downside is the exytra number of unknowns, see later.

BASIS: Let $M^1 = M^0 + \widetilde{M}^1$ with $w(x,t) = \overline{w}_i(t) + \widetilde{w}_i(t)\varphi_i(x)$ with $\varphi_i(x) = 2h_i^{-1}(x - x_i)$.

Weak formulation with test fns $v = 1\!\!1_{\mathscr{C}_i} \in M^0$ and $v = \varphi_i \in \widetilde{M}^1$:

$$h_i \frac{d\overline{w}_i(t)}{dt} = f^*_{i-1/2}(t) - f^*_{i+1/2}(t) + \int_{\mathscr{C}_i} g(w(x,t))\,dx\,,$$

$$\int_{\mathscr{C}_i} \varphi_i(x)^2\,dx\,\frac{d\widetilde{w}_i(t)}{dt} = \frac{2}{h_i}\int_{\mathscr{C}_i} f(w(x,t))\,dx - f^*_{i-1/2}(t) - f^*_{i+1/2}(t) + \int_{\mathscr{C}_i} g(w(x,t))\varphi_i(x)\,dx\,.$$

Apply Midpt quadrature to integrals with Simpson's or Trap. Rule for $\int \varphi_i(x)^2$. hence weight $\frac{1}{3\kappa}h_i$

$$\frac{d\overline{w}_i(t)}{dt} = \frac{1}{h_i}\Big(f^*_{i-1/2}(t) - f^*_{i+1/2}(t)\Big) + g(\overline{w}_i(t))\,,$$

$$\frac{d\widetilde{w}_i(t)}{dt} = \frac{-3\kappa}{h_i}\Big(f^*_{i-1/2}(t) - 2f(\overline{w}_i(t)) + f^*_{i+1/2}(t)\Big) \;\; \kappa > 0.$$

Reaction term g is only evaluated in the mean values $\overline{w}_i$,

EFFICIENT TIME INTEGRATION METHODS

Let $w(t) \in \mathbb{R}^{2m}$ be vector all $\overline{w}_i(t)$ and $\widetilde{w}_i(t)$ values, with m = no. of grid pts

$$\frac{d}{dt} w(t) = F(w(t)) + G(w(t))$$

with $F, G : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ are convection and reaction terms.

Let $t_n = n\tau$ with step $\tau > 0$ let w_n be fully discrete approx to $w(t_n)$.

Consider the following implicit reaction -explicit convection BDF2 type scheme

$$w_n = \tfrac{4}{3} w_{n-1} - \tfrac{1}{3} w_{n-2} + \tfrac{2}{3}\tau \Big(F(2w_{n-1} - w_{n-2}) + G(w_n) \Big) ,$$

Let $w_0 = w(0)$, and $w_1 \approx w(t_1)$ computed with implicit-explicit Euler

$$w_1 = w_0 + \tau F(w_0) + \tau G(w_1) .$$

Second order accurate in time, Crouzeix and Varah type method, allows arbitrary stiffness of reaction term. Stability depends on explicit part.

DG Linear Advection $u_t + u_x = 0$, with $0 \le x \le L$ and $0 \le t \le T$, and $h = L/m$ and $x_i = (i - \frac{1}{2})h$ DG^1 scheme without limiting reads

$$\frac{d}{dt}\overline{w}_i(t) = \frac{1}{h}\Big(\big(\overline{w}_{i-1}(t) + \widetilde{w}_{i-1}(t)\big) - \big(\overline{w}_i(t) + \widetilde{w}_i(t)\big)\Big),$$

$$\frac{d}{dt}\widetilde{w}_i(t) = \frac{-3\kappa}{h}\Big(\big(\overline{w}_{i-1}(t) + \widetilde{w}_{i-1}(t)\big) - 2\overline{w}_i(t) + \big(\overline{w}_i(t) + \widetilde{w}_i(t)\big)\Big).$$

LHS boundary periodic or Dirichlet condition $\overline{w}_0(t) + \widetilde{w}_0(t) = \gamma_0(t)$,

Separating $\overline{w}_i$ and $\widetilde{w}_i$ terms gives

$$\frac{d}{dt}\overline{w}_i = \frac{1}{h}\big(\overline{w}_{i-1} - \overline{w}_i\big) + \frac{1}{h}\big(\widetilde{w}_{i-1} - \widetilde{w}_i\big),$$

$$\frac{d}{dt}\widetilde{w}_i = \frac{-3\kappa}{h}\big(\overline{w}_{i-1} - \overline{w}_i\big) - \frac{3\kappa}{h}\big(\widetilde{w}_{i-1} + \widetilde{w}_i\big).$$

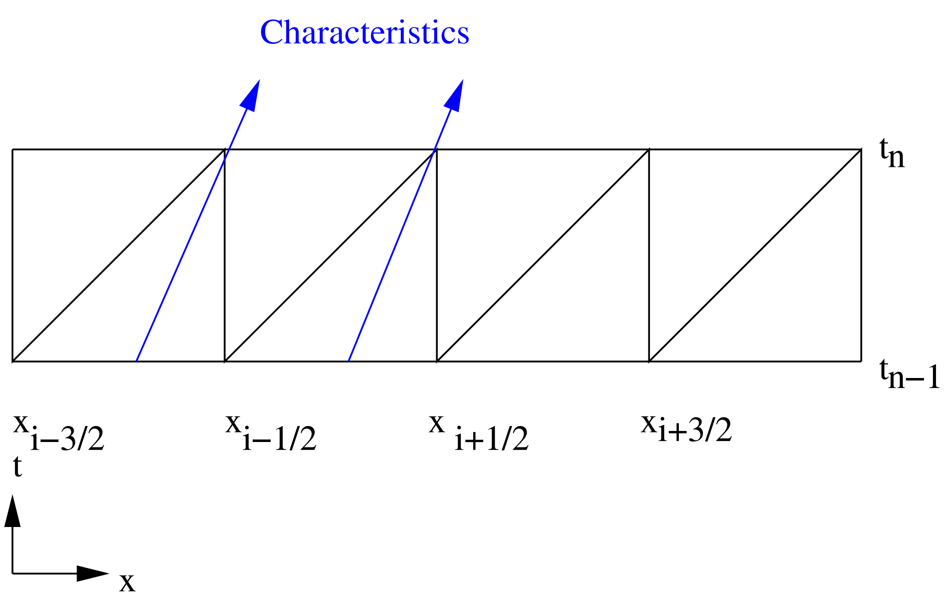
An Alternative Approach: Linear DG SPACE-TIME Characteristic METHODS:
Johnson and Pitkaranta Math Comp 86, 173, 1-26

DG Linear Advection $u_t + \gamma u_x = 0\,,$

with $0 \leq x \leq L$ and $0 \leq t \leq T$, and $h = L/m$ and $x_i = (i - \frac{1}{2})h$

Exact solution satifies $u(x,t) = u(x - \gamma(t - t_{n-1}), t_{n-1})$

Consider space-time mesh and integrate over space time triangles $\mathcal{T}_i$:



PIECEWISE LINEAR DISCONTINUOUS FNS $w = u_h, \forall\, v \in M^1$

$$\int_{\mathscr{T}_i} [w_t(x,t) + \gamma w_x(x,t)]\, v(x)\, dx - \int_{\delta \mathscr{T}_i} v(x)[u^+_{i-1/2}(t) - u^-_{i-1/2}(t)] dx = 0$$

Chose $v(x,t) = w(x - \gamma t)$ so $v(x,t)$ satisfies pde and integrate by parts:

$$\int_{x_{i-1/2}}^{x_{i+1/2}} w(x - \gamma t)[u^-(x,t_n) - u^-(x - \gamma h, t_{n-1})] dx = 0$$

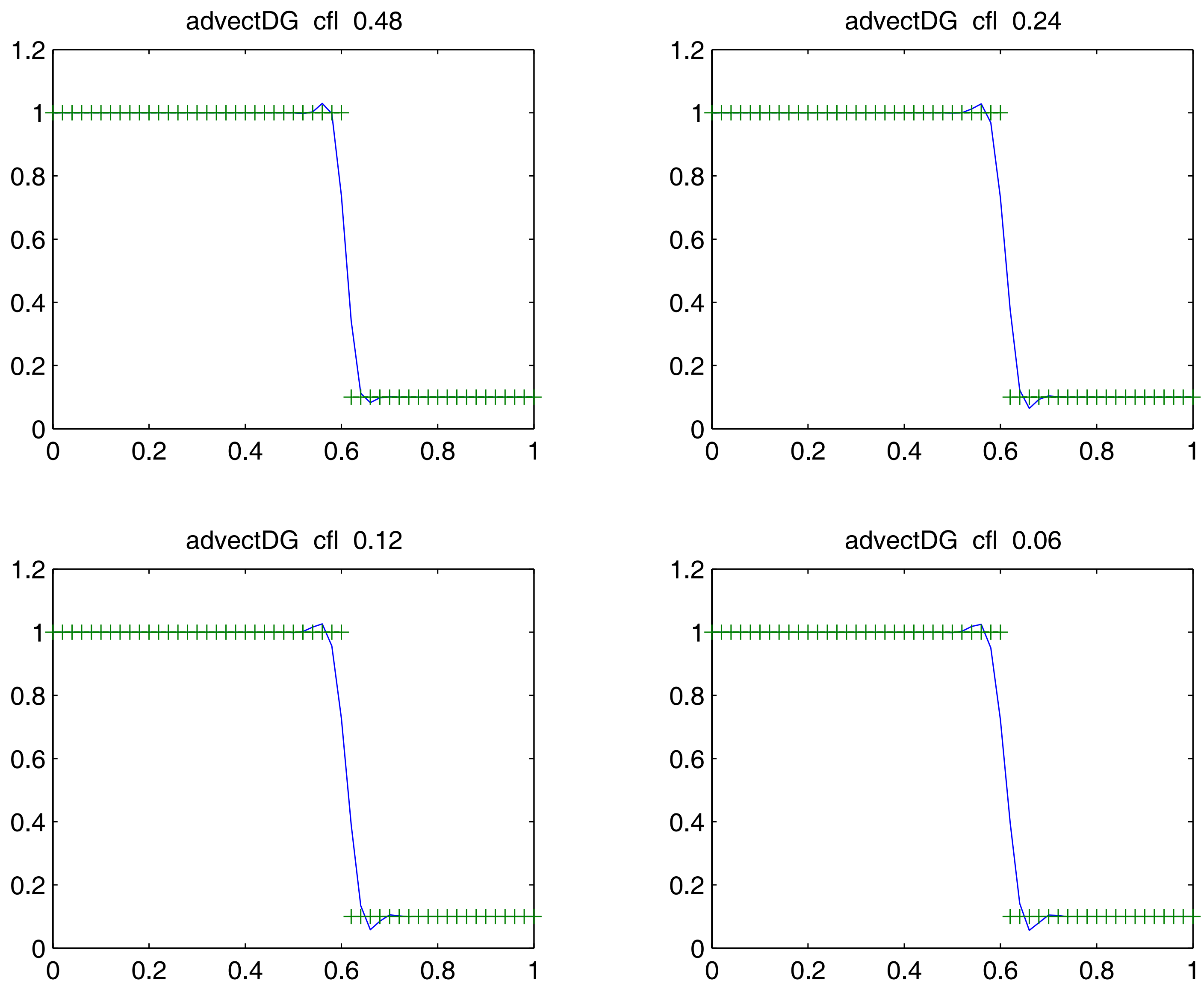
where $w(x)$ has same degree as DG poly.

$$\overline{w}_i^n = (1-\gamma)\left(\overline{w}_i^{n-1} - 0.5\gamma \widetilde{w}_i^{n-1}\right) + \gamma\left(\overline{w}_{i-1}^{n-1} + 0.5(1-\gamma)\widetilde{w}_{i-1}^{n-1}\right),$$

$$\widetilde{w}_i^n = (1-\gamma)\left(6\gamma \overline{w}_i^{n-1} + (1 - 2\gamma - 2\gamma^2)\widetilde{w}_i^{n-1}\right) + \gamma\left(-6(1-\gamma)\overline{w}_{i-1}^{n-1} - (3 - 6\gamma + 2\gamma^2)\widetilde{w}_{i-1}^{n-1}\right),$$

Third order at mesh point if solution smooth.

JOHNSON DG SPACE-TIME ADVECTION OF STEP



VON NEUMANN STABILITY

Assuming periodicity in space and $w'(t) = Aw(t)$ in $\mathbb{R}^{2m}$, Fourier transformation gives $\widehat{w}'(t) = \widehat{A}\widehat{w}(t)$ in $\mathbb{C}^2$ with

$$\widehat{A} = \frac{1}{h} \begin{pmatrix} e^{i\phi_k} - 1 & e^{i\phi_k} - 1 \\ -3\kappa(e^{i\phi_k} - 1) & -3\kappa(e^{i\phi_k} + 1) \end{pmatrix},$$

where $\phi_k = 2\pi kh$ $(k = 1, 2, \ldots, m = 1/h)$, $i = \sqrt{-1}$.

Eigenvalues of matrix $\widehat{A}$ must have a nonpositive real part. Eigenvalues λ are:

$$h\lambda_{\pm} = \tfrac{1}{2}(e^{i\phi_k} - 1) - \tfrac{3}{2}\kappa(e^{i\phi_k} + 1) \pm \sqrt{\tfrac{1}{4}((e^{i\phi_k} - 1) - 3\kappa(e^{i\phi_k} + 1))^2 + 6\kappa(e^{i\phi_k} - 1)}\,,$$

For small k, the λ_{-} eigenvalues are assoc. with the eqn for $\widetilde{w}_i$ and will damp errors faster.

Plot eigenvalues with stars for λ_+ and circles for the λ_- eigenvalues.

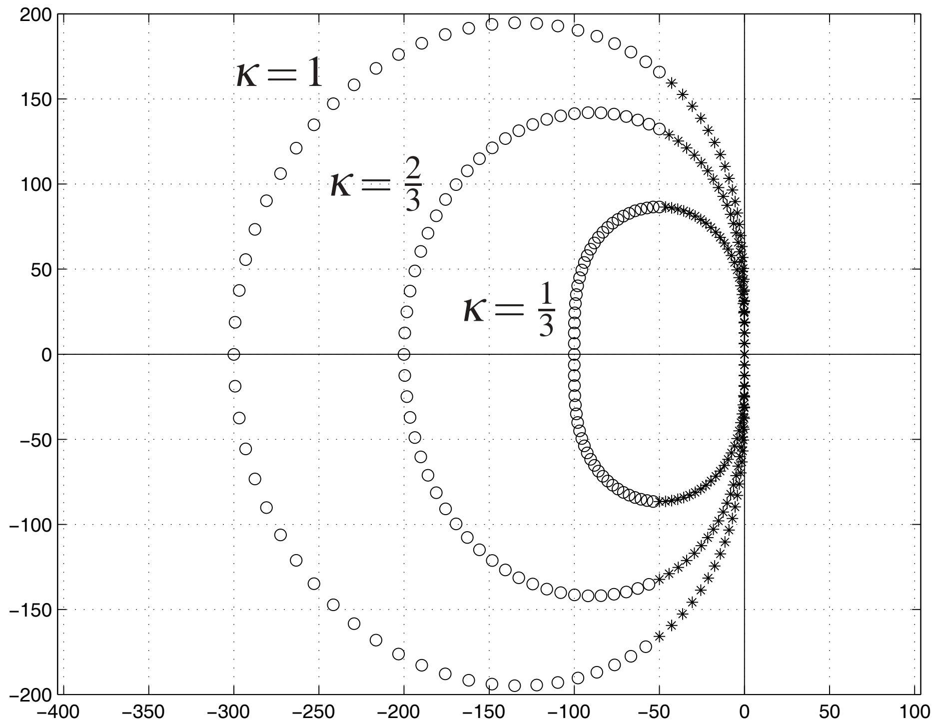


Figure 1: Eigenvalues with $h = 1/50$ for $\kappa = 1$ (outer), $\kappa = 2/3$ (middle) and $\kappa = 1/3$ (inner).

SPATIAL ERROR EQUATIONS

Exact values $\overline{u}_i(t), \widetilde{u}_i(t)$, with exact PDE solution $u(x,t)$, satisfy

$$\frac{d}{dt}\overline{u}_i = \frac{1}{h}\left(\overline{u}_{i-1} - \overline{u}_i\right) + \frac{1}{h}\left(\widetilde{u}_{i-1} - \widetilde{u}_i\right) + \overline{\delta}_i\,,$$

$$\frac{d}{dt}\widetilde{u}_i = \frac{-3\kappa}{h}\left(\overline{u}_{i-1} - \overline{u}_i\right) - \frac{3\kappa}{h}\left(\widetilde{u}_{i-1} + \widetilde{u}_i\right) + \widetilde{\delta}_i\,,$$

with (spatial) truncation errors $\overline{\delta}_i(t)$ and $\widetilde{\delta}_i(t)$ - found by Taylor series .

spatial discretization errors $\overline{\varepsilon}_i(t) = \overline{u}_i(t) - \overline{w}_i(t), \widetilde{\varepsilon}_i(t) = \widetilde{u}_i(t) - \widetilde{w}_i(t)$

Subtraction of exact from approx eqns gives

$$\frac{d}{dt}\overline{\varepsilon}_i = \frac{1}{h}\left(\overline{\varepsilon}_{i-1} - \overline{\varepsilon}_i\right) + \frac{1}{h}\left(\widetilde{\varepsilon}_{i-1} - \widetilde{\varepsilon}_i\right) + \overline{\delta}_i\,,$$

$$\frac{d}{dt}\widetilde{\varepsilon}_i = \frac{-3\kappa}{h}\left(\overline{\varepsilon}_{i-1} - \overline{\varepsilon}_i\right) - \frac{3\kappa}{h}\left(\widetilde{\varepsilon}_{i-1} + \widetilde{\varepsilon}_i\right) + \widetilde{\delta}_i\,.$$

EVOLUTION EQUATION FOR ERROR

$$\varepsilon'(t) = A\varepsilon(t) + \delta(t)$$

with $\varepsilon = (\overline{\varepsilon}_1, \cdots, \overline{\varepsilon}_m, \widetilde{\varepsilon}_1, \cdots, \widetilde{\varepsilon}_m)^T$, $\delta = (\overline{\delta}_1, \cdots, \overline{\delta}_m, \widetilde{\delta}_1, \cdots, \widetilde{\delta}_m)^T$ in $\mathbb{R}^{2m}$,

$$\varepsilon(t) = \exp\big(tA\big)\varepsilon(0) + \int_0^t \exp\big((t-s)A\big)\delta(s)\, ds\,.$$

STABILITY THEORY: $\|\exp(tA)\| \le C$ for all $t \ge 0$. Hence

$$\|\varepsilon(t)\| \le C\|\varepsilon(0)\| + Ct \max_{0\le s\le t} \|\delta(s)\|\,.$$

If $\varepsilon(0) = 0$ and $\|\delta(t)\| \le Ch^p$, then $\|\varepsilon(t)\| \le Kh^p$ on $[0,T]$

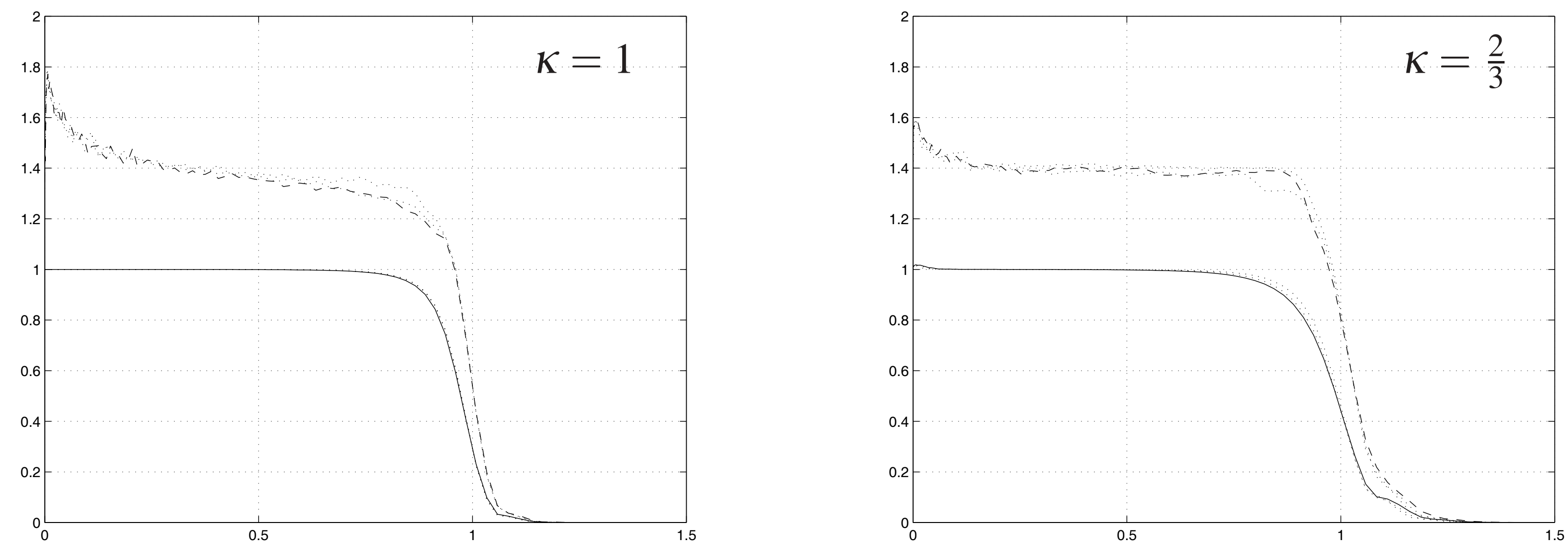


Figure 2: Example of stability bounds with $m = 50$ on random grids. Plots of $\|\exp(tA)\|$ as function of t in max-norm (dashed) and L_2-norm (solid). Dotted lines indicate repeated experiments. Left picture for $\kappa = 1$, right picture $\kappa = 2/3$.

CONSISTENCY AND FIRST ORDER TRUNCATION ERRORS

Exact solution values: $\overline{u}_i(t) = \frac{1}{h}\int_{\mathscr{C}_i} u(x,t)dx\,,\qquad \widetilde{u}_i(t) = \frac{3}{h}\int_{\mathscr{C}_i} \varphi_i(x)u(x,t)dx\,,$

$\overline{u}_i(t)\,\mathbb{1}_{\mathscr{C}_i}(x) + \widetilde{u}_i(t)\varphi_i(x)$ is L_2-projection of $u(x,t)$ on M^1.

Hence

$$\overline{u}_i(t) = u(x_i,t) + \frac{1}{24}h^2 u_{xx}(x_i,t) + \cdots\,,$$

$$\widetilde{u}_i(t) = \frac{1}{2}hu_x(x_i,t) + \frac{1}{80}h^3 u_{xxx}(x_i,t) + \cdots\,,$$

Could have used point values $u(x_i,t)$ and $\frac{1}{2}hu_x(x_i,t)$, e.g. Finite diff.

Truncation errors INCORRECTLY indicate first order scheme:

$$\overline{\delta}_i(t) = -\frac{1}{12}h^2 u_{xxx}(x_i,t) + \mathscr{O}(h^3)\,,$$

$$\widetilde{\delta}_i(t) = -\frac{1}{2}hu_{xx}(x_i,t) + \frac{1}{4}\kappa h^2 u_{xxx}(x_i,t) + \mathscr{O}(h^3)\,.$$

IMPROVED ERROR ESTIMATES

Truncation errors don't reflect observed errors. Hunsdorfer and Jaffre consider errors in more detail and show that should regard

$$\overline{u}_i(t) = u(x_i,t) + \mathscr{O}(h^2)\,, \qquad \widetilde{u}_i(t) = \frac{1}{2}hu_x(x_i,t) + \frac{1}{12\kappa}h^2u_{xx}(x_i,t) + \mathscr{O}(h^3)\,.$$

Then, wrt these solution values truncation error is

$$\overline{\delta}_i(t) = -\frac{1}{12}\left(1-\frac{1}{\kappa}\right)h^2u_{xxx}(x_i,t) + \mathscr{O}(h^3)\,, \quad \widetilde{\delta}_i = \mathscr{O}(h^2)\,.$$

Thus we see that we do indeed have second order convergence. For $\kappa = 1$ same idea shows third-order convergence of the mean values $\overline{w}_i$

THEOREM DG[1] spatial discretization with given norm (assuming stability) is order 2 for any $\kappa > 0$. $\kappa = 1$ gives order 3 for the mean values. Proof Let $\overline{u}_i(t) = u(x_i,t) + \xi h^2 u_{xx}(x_i,t)$ $\widetilde{u}_i(t) = \frac{1}{2} h u_x(x_i,t) + \eta h^2 u_{xx}(x_i,t) + \zeta h^3 u_{xxx}(x_i,t)$, consts ξ, η, ζ to be specified later. Then,

$$\frac{1}{h}\big(\overline{u}_{i-1} - \overline{u}_i\big) = -u_x + \tfrac{1}{2} h u_{xx} - \big(\tfrac{1}{6} + \xi\big) h^2 u_{xxx} + \cdots ,$$

$$\frac{1}{h}\big(\widetilde{u}_{i-1} - \widetilde{u}_i\big) = -\tfrac{1}{2} h u_{xx} + \big(\tfrac{1}{4} - \eta\big) h^2 u_{xxx} + \cdots ,$$

$$\frac{1}{h}\big(\widetilde{u}_{i-1} + \widetilde{u}_i\big) = u_x + \big(2\eta - \tfrac{1}{2}\big) h u_{xx} + \big(2\zeta - \eta + \tfrac{1}{4}\big) h^2 u_{xxx} + \cdots .$$

Using $u_t + u_x = 0$ to swop space and time derivs gives:

$$\overline{\delta}_i = \big(\eta - \tfrac{1}{12}\big) h^2 u_{xxx} + \mathscr{O}(h^3) ,$$

$$\widetilde{\delta}_i = \big(6\kappa\eta - \tfrac{1}{2}\big) h u_{xx} + \Big(\tfrac{1}{4}\kappa - (1+3\kappa)\eta + 3\kappa(2\zeta - \xi)\Big) h^2 u_{xxx} + \mathscr{O}(h^3) .$$

Oder 2 $\kappa > 0$ Order 3 $\kappa = 1, \eta = 1/12$

DG METHOD - ACCURACY ON SMOOTH ADVECTION

$u_t + u_x = 0\,, u(x,t) = \sin^2\left(\pi(x-t)\right)\,, \qquad 0 \le t \le \frac{1}{2}\,, 0 \le x \le 2\,.$

L_2-error norms for $\kappa = \frac{1}{3}, \frac{2}{3}, 1.$

mean value errors $\|\bar{\varepsilon}\|_2$ and the global errors $\|\varepsilon\|_2$,

$$\|\bar{\varepsilon}\|_2 = \Big(\sum_{j=1}^{m} h_j|\bar{\varepsilon}_j|^2\Big)^{1/2}, \qquad \|\varepsilon\|_2 = \Big(\sum_{j=1}^{m} h_j|\bar{\varepsilon}_j|^2 + \tfrac{1}{3}h_j|\widetilde{\varepsilon}_j|^2\Big)^{1/2}.$$

Errors plotted wrt $1/m$ - number of randomly chosen grid points.

The results on random grids similar to uniform grids. Third order convergence if $\kappa = 1$.

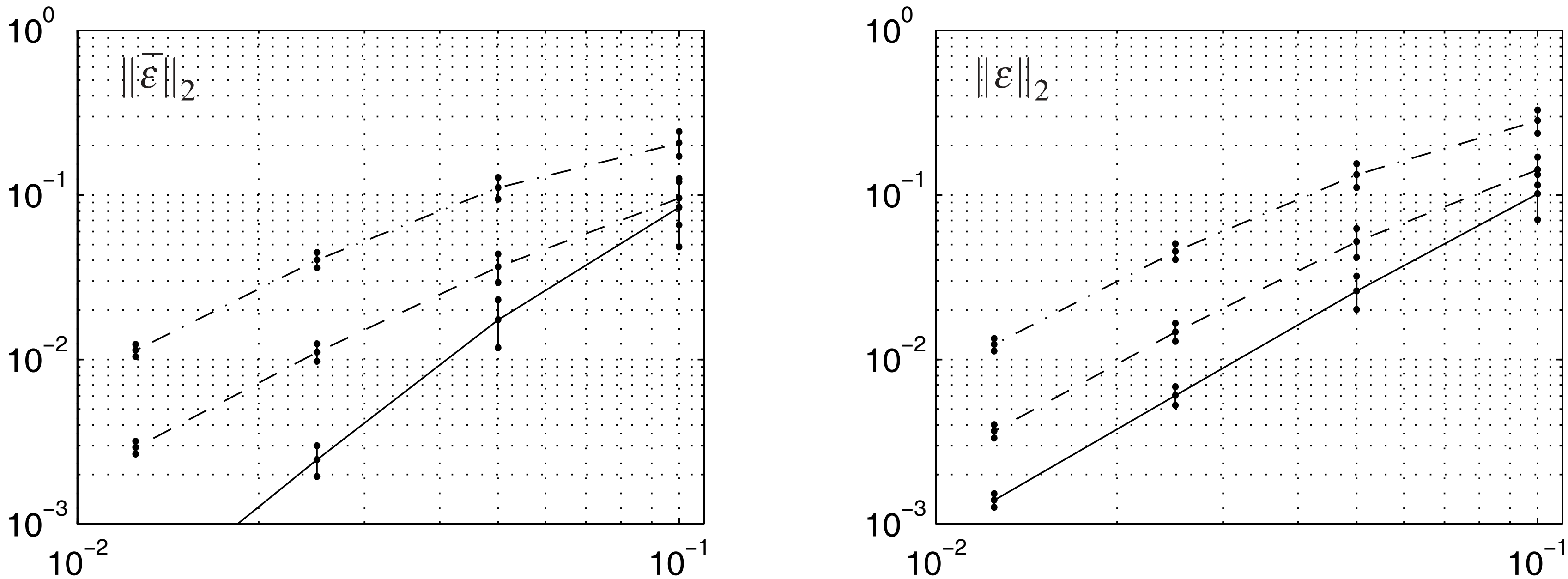


Figure 3: L_2-errors versus $1/m$ with mean-values (left) and global solution (right) for linear convection on random grids. Results for $\kappa = 1$ (solid), $\kappa = 2/3$ (dashed) and $\kappa = 1/3$ (dash-dot).

ACCURACY DECREASE USING LIMITERS WITH DG METHODS

van Leer Limiter small time steps - spatial error dominates.

given $\overline{w}_i, \widetilde{w}_i$ at time level t_n, adjust $\widetilde{w}_i$

$$\widetilde{w}_i := \mathscr{M}(\widetilde{w}_i, \overline{w}_{i+1} - \overline{w}_i, \overline{w}_i - \overline{w}_{i-1})$$

with minmod fn

$$\mathscr{M}(a,b,c) = \begin{cases} \text{sign(a)}\min(|a|,|b|,|c|) & \text{if sign}(a) = \text{sign}(b) = \text{sign}(c), \\ 0 & \text{otherwise.} \end{cases}$$

Very little difference between κ values - accuracy is determined by limiter.

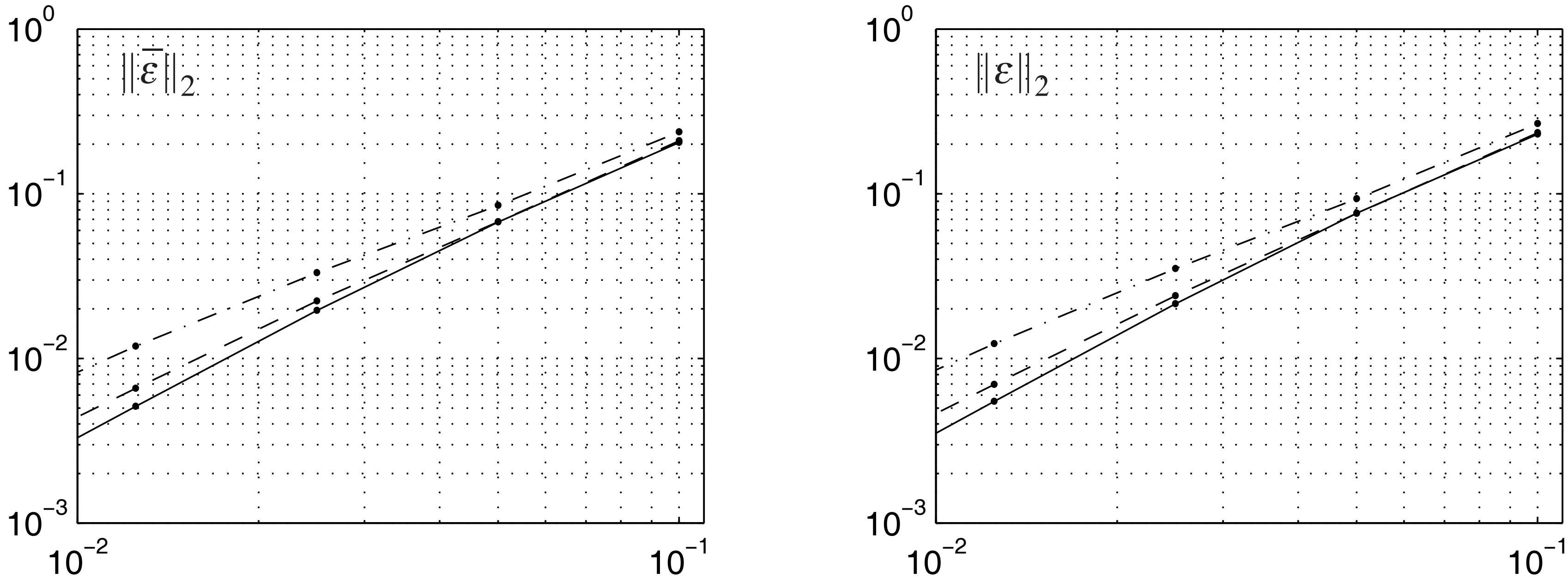


Figure 4: L_2-errors versus $1/m$ with mean-values (left) and global solution (right) for linear convection, uniform grids with limiting. Results for $\kappa = 1$ (solid), $\kappa = 2/3$ (dashed) and $\kappa = 1/3$ (dash-dot).

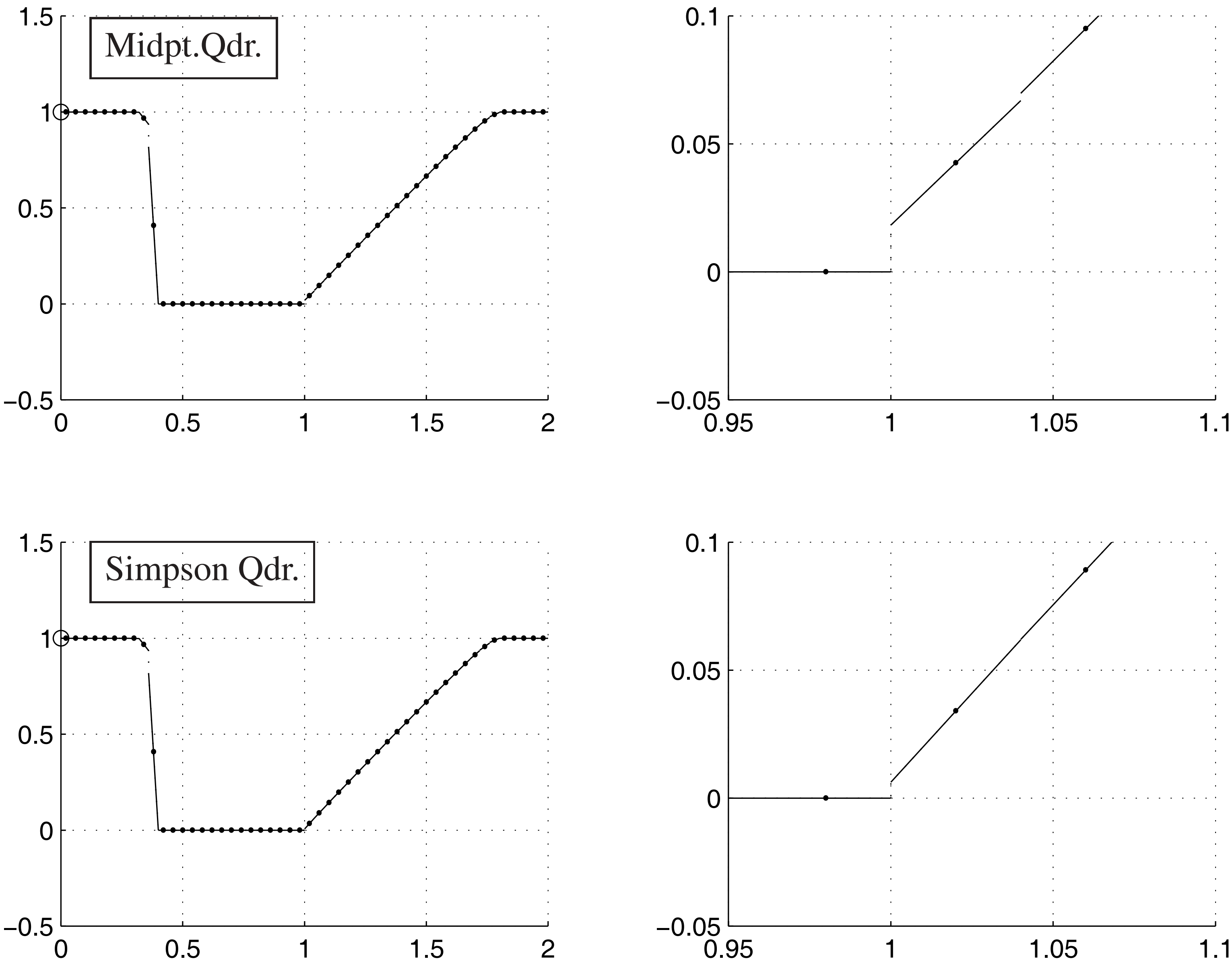
BURGERS EQUATION WITH MISPOINT AND SIMPSON RULES

$$u_t = f(u)_x\,, \qquad f(u) = \frac{3}{4}u^2\,,$$

on $0 \leq t \leq T = \frac{1}{2}$, $0 \leq x \leq L = 2$, with $u_0(x) = 0$ for $0 < x \leq \frac{1}{2}$ and $u_0(x) = 1$ elsewhere, and with $u(0,t) = 1$. Uniform grid with $h = 1/25$.

Results at $T = \frac{1}{2}$ for the slope limited DG^1 scheme with midpoint quadrature and with Simpson quadrature.

As expected Simpson quadrature only gives slightly better results near $f'(u) = 0$.


Midpt.Qdr.
Simpson Qdr.
1.5
1
0.5
0
−0.5
0.5
1.5
2
0.1
0.05
−0.05
0.95
1.05
1.1

TIME INTEGRATION METHODS BDF2 vs TVD 3-step

BDF2 type scheme for convection : $\frac{d}{dt}w(t) = F(w(t))$ is

$w_n = \frac{4}{3}w_{n-1} - \frac{1}{3}w_{n-2} + \frac{2}{3}\tau F(2w_{n-1} - w_{n-2})$ $\tau > 0$ is time step

3-step 2nd order TVD3 scheme of Shu $w_n = \frac{3}{4}w_{n-1} + \frac{1}{4}w_{n-3} + \frac{3}{2}\tau F(w_{n-1})$

	(BDF2)	(TVD3)
$\kappa = \frac{1}{3}$	0.44	0.35
$\kappa = \frac{2}{3}$	0.27	0.20
$\kappa = 1$	0.20	0.14

Table 1: Maximal Courant numbers for stability of BDF2 and TVD3 methods

TOTAL VARIATION DIMINISHING PROPERTIES

For $v \in M^1$ total variation of mean values $\overline{v}_j$ is $|v|_{\overline{TV}} = \sum_{j=1}^{m-1} |\overline{v}_j - \overline{v}_{j+1}|$.

Time method is called TVD (total variation diminishing) if $|w_n|_{\overline{TV}} \le |w_0|_{\overline{TV}}$.

Theorem The DG 1 method defined above is TVD under the restriction $\nu \le \frac{1}{4}$.
The combination $w_1 = w_0 + \tau F(w_0)$, and the explicit trapezoidal rule

$$w_1^* = w_0 + \tau F(w_0), \qquad w_1 = w_0 + \frac{1}{2}\tau F(w_0) + \frac{1}{2}\tau F(w_1^*).$$

is TVD for $\nu \le \frac{5}{16}$. **Proof** See Hunsdorfer and Jaffre. BDF2 type method slightly better than TVD3 wrt stability and TVD and combines with implicit BDF2 to treat stiff terms,

ADSORPTION-DESORPTION TEST – ACCURACY VERSUS CPU

External flow field q unknown dissolved concentration u and adsorbed concentration v. system of convection-reaction equations

$$u_t + (qu)_x = k\big(v - \psi(u)\big)\,, k = 1000, k_1 = k_2 = 100$$

$$v_t = -k\big(v - \psi(u)\big), \psi(u) = k_1 u/(1 + k_2 u)$$

where $k > 0$ is the reaction rate and $\psi(u)$ = steady state ratio u and v, velocity q is

$$q(t) = \begin{cases} 1 & \text{if } t \leq 1\,, \\ -1 & \text{if } t > 1\,, \end{cases}$$

Initial and boundary conditions: $u, v \equiv 0$, $u(0,t) = 1$ for $t \leq 1$, $u(1,t) = 0$ for $t > 1$. Non-continuous solution for $t < 1$ gives $O(h^1)$ convergence.

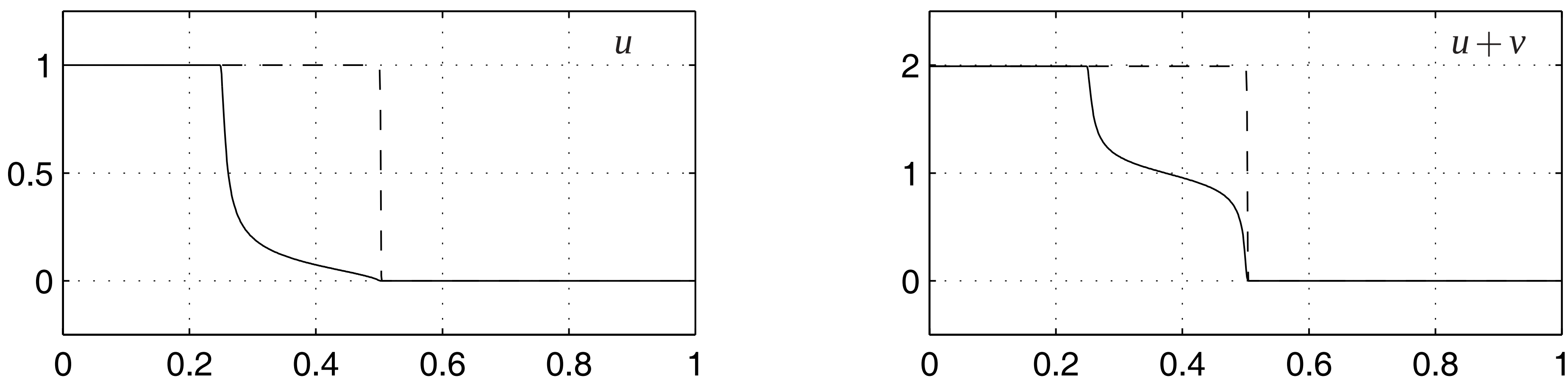


Figure 6: Adsorption-desorption test. Plots of dissolved concentration u (left) and total concentration $u+v$ (right) at time $t=1$ (dashed) and $t=\frac{5}{4}$ (solid).

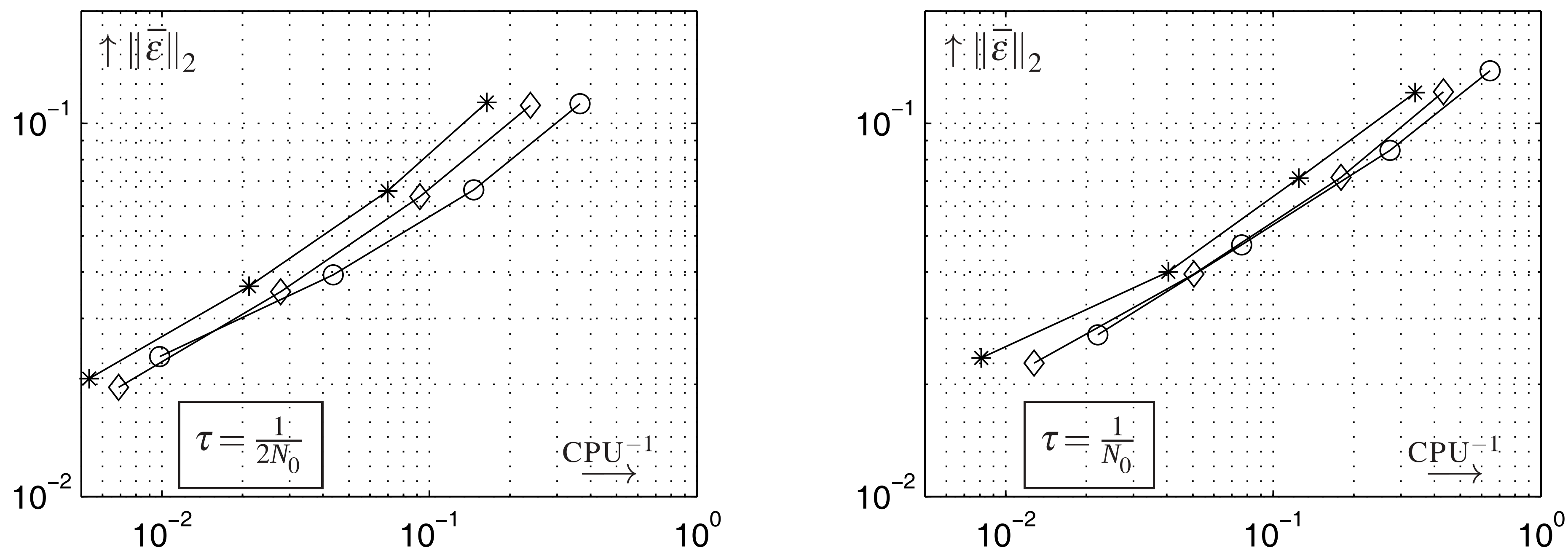


Figure 7: Mean errors $\|\bar{\varepsilon}\|_2$ of total concentration $u+v$ at output time $t = 5/4$ as function of the inverse of CPU timings for $\kappa = 1$ ($*$-marks), $\kappa = 2/3$ ($\diamond$-marks) and $\kappa = 1/3$ ($\circ$-marks). Left plot with $\tau = 1/(2N_0)$, right plot with $\tau = 1/N_0$. $\kappa = \frac{1}{3}$ gives larger spatial errors than $\kappa = \frac{2}{3}, 1$, but uses larger timesteps. $\kappa = 1$ requires twice the CPU time of $\kappa = \frac{2}{3}$ to achieve a certain accuracy level.

CONCLUSIONS AND CURRENT RESEARCH ACTIVITY

Low order quadrature OK, Diffusion can betreated in the same framework

smooth solution and no limiter $\kappa = 1$ non-smooth limited $\kappa = \frac{1}{3}$ better efficiency

Current work by Suli Houston Flaherty Cockburn etc etc on:

Error analysis and estmation variable h-p adaptation

Adjoint Methods, benchmark calculations (See Flaherty et al.)

ONLY DOWNSIDE IS HIGH COST COMPARED TO CG METHODS: Linear (4X to 20X) Quadratric (2x to 7x) assuming linear complexity.