

MJ11 DEPENDENT NESTED LOOPS FORMULA SHEET 2009

In the case where we have dependent nested loops the analysis of a program can often be more complicated.

Consider the following piece of code:

```
for i ← 1 to n do
  for j ← 1 to i do
    ...
```

In this example the inner loop is executed once for the first iteration ($i = 1$ therefore j starts at 1 and finishes at 1), twice for the second iteration ($i = 2$ therefore j starts at 1 and finishes at 2), three times for the third iteration ($i = 3$ therefore j starts at 1 and finishes at 3) and so on until the last iteration when the inner loop is executed n times ($i = n$ therefore j starts at 1 and finishes at n). So each operation inside the inner loop is performed

$$1 + 2 + 3 + 4 + \dots + (n - 1) + n \quad \text{times.}$$

There is a formula to calculate the sum of increasing integers, it is as follows. Let f and l be two positive integers. Then if $f < l$ the sum of the integers from f to l is as follows

$$f + (f + 1) + (f + 2) + \dots + (l - 1) + l = \frac{1}{2}(f + l)(l - f + 1)$$

in words this is the same as

$$= \frac{1}{2}(\text{first term in sequence} + \text{last term in sequence})(\text{number of terms in sequence}).$$

So for the example above we have

$$1 + 2 + 3 + 4 + \dots + (n - 1) + n = \frac{1}{2}(1 + n)(n).$$

Since 1 is the first in the sequence, n is the last thing and in total there are n integers in the list.

EXAMPLES

1. $3 + 4 + \dots + 10 = \frac{1}{2}(3 + 10)(8) = \frac{1}{2} * 104 = 52$, since 3 is the first in the sequence, 10 is the last thing in the sequence and in total there are $10 - 3 + 1 = 8$ integers in the list.
2. $1 + 2 + 3 + 4 + \dots + (n - 1) = \frac{1}{2}(1 + n - 1)(n - 1) = \frac{1}{2}(n)(n - 1)$, since 1 is the first in the sequence, $n - 1$ is the last thing in the sequence and in total there are $n - 1$ integers in the list.
3. $2 + 3 + 4 + \dots + (n - 1) + n = \frac{1}{2}(2 + n)(n - 1)$, since 2 is the first in the sequence, n is the last thing in the sequence and in total there are $n - 2 + 1 = n - 1$ integers in the list.