Convexity in Discrete Space

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Abstract. This paper looks at axioms for convexity, and shows how they can be applied to discrete spaces. Two structures for a discrete geometry are considered: oriented matroids, and cell complexes. Oriented matroids are shown to have a structure which naturally satisfies the axioms for being a convex geometry. Cell complexes are shown to give rise to various different notions of convexity, one of which satisfies the convexity axioms, but the others also provide valid notions of convexity in particular contexts. Finally, algorithms are investigated to validate the sets of a matroid, and to compute the convex hull of a subset of an oriented matroid.

Key Words: Convexity axioms, alignment spaces, affine spaces, convex spaces, convex hull, discrete geometry, oriented matroids, cell complexes, matroid algorithms.

1 Introduction

The concept of a convex region plays an important part in many practical computations in GIS. It is also a fundamental component in the applications of spatial information theory in other areas. As Boissonnat and Yvinec note [BY98, p125]: "Convexity is a fundamental notion for computational geometry, at the core of many computer engineering applications, for instance in robotics, computer graphics, or optimization." Convexity has been studied both in point-based quantitative theories of space, based on numerical coordinates, and also in the context of qualitative spatial reasoning. As an example of the latter, we note that the region-connection calculus was extended [CB+97, p287] by adding a one place function which returned the convex hull of a region.

There has recently been much interest within the spatial information theory community in discrete notions of space, e.g. [Win95,Gal99,MV99,Ste00,RS02]. If discrete space is to provide an adequate foundation for the representation of geographic information, it will need to be equipped with a notion of convexity. In this paper we examine how convexity may be defined in discrete space, and how it is possible to provide algorithms to compute the convex hull of a discrete region.

The traditional definitions of convexity (in terms of line segments) in Euclidean space are not applicable to discrete space, so it is necessary to investigate a more abstract axiomatic formulation, in terms of alignment axioms. This we do in Section 2.1. We then look at aligned spaces, and show that we can have discrete models of aligned spaces that hold the anti-exchange axiom, and further models that hold the exchange axiom. These correspond to convex geometries and affine geometries respectively, the latter giving rise to matroids. We define constructs in $\mathbb{Z}^n$ that give rise to convex and affine geometries.

We go on to investigate cell complexes, and see how these give rise to aligned spaces. Several definitions of convexity are given, and relationships between these are given. It is important that convexity is not just studied in the abstract: it is necessary to show how computations can be carried out. We provide some algorithms for the computation of convex hulls of oriented matroids in Section 5. A brief introduction to the necessary oriented matroid concepts is provided earlier in Section 3.1.

2 Axiomatizing Convexity

This section gives an axiomatization of convex and affine geometries in a very general setting. The concepts themselves come from the corresponding concepts of convex and affine geometries defined on a vector space over the real numbers, derived from linear combinations of points in a vector space. Given a set of vectors $v_0, \ldots, v_k$, and a set of scalars, $\lambda_0, \ldots, \lambda_k$, a linear combination of these vectors given the set of scalars is a vector given by:

$$\sum_{i=0}^{k} \lambda_i v_i$$

A *subspace* of a vector space is a subset $S \subset \mathbb{R}^n$ which contains all linear combinations of points in S (i.e. closed under linear combinations). Examples include $\mathbb{R}^n$, the origin $\mathbf{0}$, and any line passing through the origin. Note that the origin will always be in a subspace, since all scalars can be set to zero.

Affine combinations are restricted linear combinations; the scalars λ_i must add up to exactly 1. This gives rise to the notion of affine sets, the sets $S \subset \mathbb{R}^n$ closed under affine combinations. The affine sets in $\mathbb{R}^n$ include all translations of the set of subspaces. An affine set is often characterized as containing all of the lines through each pair of its points.

Convex combinations are restricted affine combinations; not only must the scalars add up to 1, but they must also lie in the range $[0, 1]$. Convex sets are sets closed under convex combinations and are often characterized as sets containing all line segments between each pair of its points.

It is clear from the above definitions that linear, affine and convex sets are closely related, and that the relation $Linear \subseteq Affine \subseteq Convex$ holds. The notion of aligned spaces given below is an abstraction of convex and affine geometries

2.1 Alignment Spaces

The concept of alignment spaces was introduced by Coppel[Cop98] as a general axiomatization of convexity. The main focus of his work is on continuous settings, but we show here how the axioms of alignment spaces can be successfully applied to discrete spaces.

In Euclidean space a set, S, is said to be convex if for every pair of points in S the straight line between them is also in S. We can show that this definition does not directly extend to discrete space. If we define a straight line between two points, a and b, in $\mathbb{Z}^n$ as being the set of all points collinear with those points, and between them. That is, $line(a, b) = \{x \in \mathbb{Z}^n | x = \lambda a + \gamma b,\ \lambda + \gamma = 1,\ 0 \leq \lambda, \gamma\}$. Figure 1 shows that such a definition of convexity does not fit with an intuitive notion of convex set. In this example, the three points alone are by the naive definition a convex set.

The following definition begins to capture some of the properties we would expect a convex set to possess.

Definition 1 (Alignment Space). *An* alignment space *is a pair $A = (X, \mathcal{C})$ where X is a set and $\mathcal{C}$ is a set of subsets of X such that the following axioms hold:*

(A0) $\varnothing \in \mathcal{C}$ *(Empty set)*
(A1) $X \in \mathcal{C}$ *(Top set)*
(A2) $\forall Z \subseteq \mathcal{C},\ Z \neq \varnothing \rightarrow \bigcap Z \in \mathcal{C}$ *(Intersection)*
(A3) $\forall Z \subseteq \mathcal{C},\ Z \neq \varnothing$ *if Z is totally ordered by inclusion, then* $\bigcup Z \in \mathcal{C}$ *(Qualified union)*

We call the sets in $\mathcal{C}$ aligned sets*. The intersection of all aligned sets containing a set $S \subseteq X$ is called the* alignment hull *of S, denoted $[S]$. This intersection always exists due to (A2) above.*

Proposition 1. *Note that the following properties hold of any alignment hull:*

H0 $[\varnothing] = \varnothing$
H1 $S \subseteq [S]$
H2 $S \subseteq T \rightarrow [S] \subseteq [T]$
H3 $[[S]] = [S]$
H4 $[S] = \bigcup\{[T] : T \subseteq S \textit{ and } T \textit{ is finite}\}$

The proof for this proposition can be found in [Cop98]

In brief the axioms state that the entire set and the empty set are aligned, arbitrary intersections of aligned sets are aligned, and the union of totally ordered sets of aligned sets are also aligned. These are properties we would expect of convex sets. These axioms alone are not enough however, as Figure 1 shows.

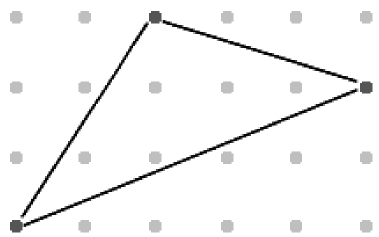


Fig. 1. A 'convex' set according to the naive definition.

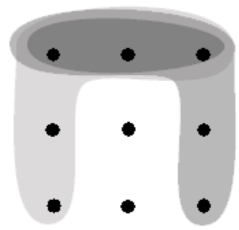


Fig. 2. A subset of 2^X that satisfies the alignment axioms.

Figure 2 shows two sets of points. Let $\mathcal{C}$ be a set containing these two sets along with their intersection, the entire space X and the empty set. Then, $\mathcal{C}$ satisfies all four axioms, and is hence an aligned space. This is an aligned set, but is not convex in our usual understanding of the concept.

2.2 Convex Geometries

To overcome the problem just mentioned, a further axiom, known as the anti-exchange property, is added.

Definition 2 (Anti-exchange Property). *The anti-exchange property is characterized by the following axiom:*

(AE) $\forall x, y \in X,\ \forall S \subseteq X,$ *if* $x \neq y$ *and* $y \in [S \cup x]$ *and* $y \notin [S]$ *then* $x \notin [S \cup y]$ *(Anti-exchange)*

Clearly the example in Figure 2 fails to satisfy this additional axiom. Take for example S to be one of the two depicted subsets, and x and y as distinct points not in S then $S \subseteq X,\ x \neq y$ as required, $y \in [S \cup x]$ since $[S \cup x] = X$ (X is the largest aligned set containing $S \cup x$), $y \notin [S]$ since S is the smallest aligned set containing itself. However, $x \in [S \cup y]$ since X is again the smallest aligned set containing the union of S and y.

We shall refer to aligned spaces for which the anti-exchange property holds as *convex geometries*, and the alignment hulls in this special case *convex hulls*. Note that the term usually refers to geometries defined with points and segments as primitives (see [Cop98]), but in this paper we have already seen that such an axiomatization would not be applicable in the case of discrete space.

Figure 3 shows an example of a space that satisfies the five axioms A0-A3 and AE. Note that though all axioms are satisfied, there are subsets missing that we would normally consider to be convex.

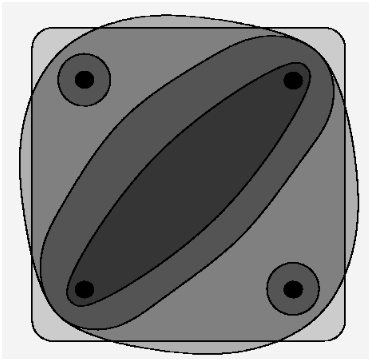


Fig. 3. A convex geometry.

2.3 Affine Geometries

The exchange property essentially says that if an element, x, is in the hull of a set $S \cup y$, then y is in the hull of the set $S \cup x$. i.e. x and y are interchangeable for purposes of generating hulls. Due to the difference between this axiom and the convexity (anti-exchange) axiom, we denote the hull in this case by angle brackets, e.g. $\langle S \rangle$.

Definition 3 (Exchange Property). *The exchange property is characterized by the following property:*

(E) $\forall x, y \in X,\ \forall S \subseteq X,$ *if* $x \neq y$ *and* $y \in \langle S \cup x \rangle$ *and* $y \notin \langle S \rangle$ *then* $x \in \langle S \cup y \rangle$ *(Exchange)*

Note that since $y \notin \langle S \rangle$ *and* $y \in \langle S \cup x \rangle$ *then* $x \notin \langle S \rangle$.

Together with (A0)–(A3), this axiom gives us the notion of an affine geometry, and the alignment hulls in this case shall be referred to as *affine hulls*. The intuition behind the notion of affine hull of a set S is that it is the smallest affine set of the given space containing S. For example, given two points in Euclidean space, the affine hull of those points is the straight line passing through them.

3 Structures for Discrete Geometry

We consider here two structures for discrete geometry; (oriented) matroids and cell complexes.

Matroids (and oriented matroids) are abstract structures comprising of a finite set of discrete objects (that is we can map the set onto a subset of the integers) and a set of subsets of that set which satisfy certain properties - i.e. a combinatorial structure. The set of axioms obeyed by the subsets of the matroid are given below, and as we shall see the resulting structure is rich enough to enable us to model many important geometric constructs such as linear independence, bases, subspaces etc. Due to the discrete nature of the structure, matroids also allow us to do computations in a robust manner - no rounding errors occur since we simply manipulating finite sets to get our results.

Cell complexes can be thought of as a partition of space into objects of different dimension. In two dimensions for example they comprise of 0-dimensional points, 1-dimensional line segments and 2-dimensional regions. A simple cell complex structure, commonly called a cartesian complex, comprises of a set of grid points taking values from $\mathbb{Z}^n$ as the 0(dimensional)-cells, the vertical and horizontal (open) line segments between those cells as the 1-cells, and the square region between 4 0-cells and 4 1-cells as the 2-cells. Other examples include simplicial complexes, and more general partitions such as the triangulation in Figure 4.

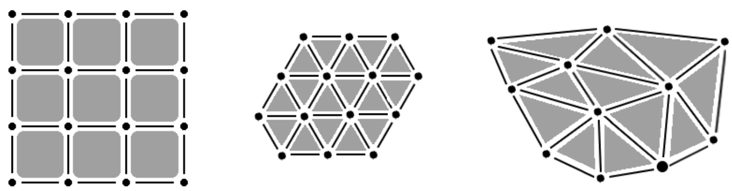


Fig. 4. Cell complexes: Cartesian Complex; Simplicial Complex; Triangulation

The following sections show that the two structures can be effectively used together to provide robust calculation with a useful topology.

3.1 Capturing the Geometry of Discrete Space with Matroids

Though matroids are essentially an abstract combinatorial structure, they have roots in linear algebra, and consequently capture a good amount of geometric structure. In fact (H1)-(H3) and (E) are the axioms of a closure operator on a matroid. In this section we give a brief introduction to independence axioms on matroids, and circuit axioms on oriented matroids (matroids with some additional structure).

Definition 4 (Matroid: Independence axioms).
A Matroid is a pair $M = (E, \mathcal{I})$ where E is a finite set, and $\mathcal{I}$ is a collection of subsets of E (called independent subsets*) satisfying the following axioms:*

(I1) $\varnothing \in \mathcal{I}$
(I2) *If $I \in \mathcal{I}$ and $I' \subseteq I$ then $I' \in \mathcal{I}$*
(I3) *If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then $\exists e \in I_2 - I_1,\ I_1 \cup \{e\} \in \mathcal{I}$*

$\mathcal{I}$ are the independent sets *of the matroid M; E is called the* ground set.

In the following sections we describe how discrete spaces can be modelled using matroids, and in particular how it is possible to construct an oriented matroid structure with which to develop a convex geometry. Some of the combinatorial structure previously mentioned will be used in these later sections, and are defined as follows:

Independents: Let $M = (E, \mathcal{I})$ be a matroid defined in terms of its independent sets. The maximal independent sets are called *bases* of the matroid.

Circuits: A set $D \subseteq E$ is called *dependent* if $D \notin \mathcal{I}$. The dependent sets which are minimal in E are called the *circuits* of the matroid.

Subspaces: The *rank* of a subset $S \subseteq E$ is the size of the largest independent set it contains; $\rho(S) = max(|X| : X \subseteq S, X \in \mathcal{I})$.

A *subspace* or *flat* of a matroid is a subset $F \subseteq E$ where for every $x \in E - F, \quad \rho(F \cup \{x\}) = \rho(F) + 1$; that is a flat is maximal with respect to its rank. The rank of the matroid M, ρM, is the rank of E. It should be noted that flats can be equivalently characterized as the fixed points of a closure operator.

A *hyperplane* is a flat whose rank is one less than the rank of E.

Co-circuits and Duality: The set theoretic complement of a hyperplane, h, in M is called a *cocircuit* of M. This is the co-circuit *associated with* h. A property of the cocircuits, $\mathcal{I}^*$, of a matroid, M, is they form the set of circuits in the dual matroid $M^* = (E, \mathcal{I}^*)$. This is a vital result which we use in constructing an oriented matroid later in this paper.

So far, the matroid structure we have is powerful enough to model certain geometric constructs, such as independent sets, flats and so on. However to enable us to talk about convexity we need a little more structure; we introduce a partition (orientation) on the circuits to get an *oriented matroid*. First we shall define the concept of *signed sets*.

Definition 5 (Signed Set). *A* signed set X *is a set* $\underline{X}$ *along with a partition* (X^+, X^-) *of that set where* $X^+ \cup X^- = \underline{X}$ *and* $X^+ \cap X^- = \varnothing$.

$\underline{X}$ *is called the* support *of* X*,* X^+ *is the set of* positive elements *and* X^- *is the set of* negative elements.

The opposite *of a signed set* $X = (X^+, X^-)$*, is* $-X = (X^-, X^+)$.

An oriented matroid has signed sets in place of ordinary sets as its circuits, and hence we must alter the axioms accordingly.

Definition 6 (Oriented Matroid: Circuit Axioms).

An oriented matroid *is a pair* $O = (E, \mathcal{C})$*, where* $\mathcal{C}$ *is a collection of signed subsets of a finite set* E *satisfying the following axioms:*

(OC1) $\varnothing \notin \mathcal{C}$

(OC2) $\mathcal{C} = -\mathcal{C}$ *(symmetry)*

(OC3) $\forall X, Y \in \mathcal{C},\ \ \textit{if } \underline{X} \subseteq \underline{Y} \textit{ then } X = Y \textit{ or } X = -Y$ *(incomparability)*

(OC4)

$$\forall X, Y \in \mathcal{C},\ X \neq -Y,\ \textit{and } e \in X^+ \cap Y^-$$
$$\exists Z \in \mathcal{C} \textit{ such that } Z^+ \subseteq (X^+ \cup Y^+) \setminus \{e\} \textit{ and}$$
$$Z^- \subseteq (X^- \cup Y^-) \setminus \{e\}$$

(weak elimination)

The oriented matroid structure gives us access to various useful constructs, such as the notion of halfspaces. The power of this additional structure will be seen in Section 4.1.

3.2 Modelling Discrete Space with Cell Complexes

Cell complexes are becoming ever more popular as an alternative model of discrete space to the standard pixel/point approaches, primarily for their topological properties. In approaches based in $\mathbb{Z}^n$, sets of points have a trivial topology, that is the open and closed sets in the topology are the same. Attempts at adding a notion of boundary to sets of points do not properly address this: problems crop up such as regions and their complements do not share a boundary, boundaries have non-zero area, and other such anomalies (see [Kov89]).

Cell complexes are an alternative to the $\mathbb{Z}^n$ based approaches, adding higher dimensional entities, to 'fill in the gaps' between points. The advantage of doing so is that cell complexes give a far richer topological structure; this allows, for example, notions of open and closed sets, boundary and connectedness.

Formally, a cell complex is a triple (C, B, dim). C is a set, B is an irreflexive, anti-symmetric and transitive bounding relation (read $(a, b) \in B$ as 'a bounds b'), and dim is a function taking elements of the C to an integer value representing the dimension of the element. The only other constraint is that if a bounds b then $dim(a) < dim(b)$.

There are many ways of modelling a discrete space with cell complexes. One example which we look at further in Section 4.2 take the points of $\mathbb{Z}^2$ as the elements of 0 dimension (0-cells), horizontal and vertical pairs of adjacent points as the 1-cells, and 'squares' of four adjacent points as the 2-cells. The bounding relation is defined in the obvious way. This structure is known as a *(2 dimensional) cartesian* cell complex (see [Web01] for further details).

4 Digital Geometries on $\mathbb{Z}^2$

Section 3 looked at two structures with which we may represent discrete space. In this section we show how convexity can be represented in these structures. In each case we give convexity operators which satisfy the axioms for a convex geometry given in Section 2.1. Interestingly, we also show that there are other convexity operators definable on cell complexes which would be appropriate in different contexts, but do not necessarily satisfy the axioms of a convex geometry.

4.1 Matroids and Convexity on $\mathbb{Z}^2$

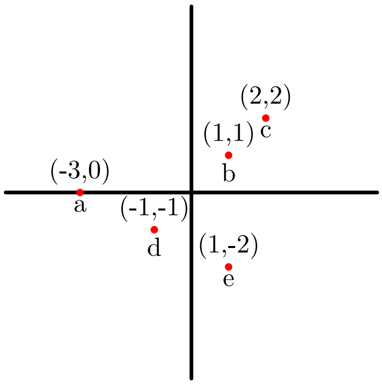
We have already established various results about convexity; now we look at how oriented matroids can be used as a model of convexity, and at the same time abstract certain properties of digital space. We begin by looking at a matroid, M, whose ground set will be a finite subset of $\mathbb{Z}^2$, and whose independent sets are the affine independent subsets of the space: $\mathit{affInd}(S) \iff \forall x, y \in S,\ x \neq y,\ \nexists z \in S\ \lambda x + \mu y = z$ for $\lambda, \mu \neq 0, \lambda + \mu = 1$. In addition, M must be of rank 3 (the largest independent sets being triples of non-collinear points), i.e. the ground set should not be equivalent to a lower dimensional subspace embedded in $\mathbb{Z}^2$.

Now, the affine sets of the space, i.e. the sets, S, for which the condition $\forall x, y, z \in S \rightarrow \exists \lambda, \mu \in \mathbb{Q},\ \lambda + \mu = 1,\ x = \lambda y + \mu z$ holds, are easily shown to be the flats of M: they are maximal sets of collinear points (hence subspaces), and the largest independent set is of rank 2 (any pair of points). Hence the set of all flats forms an affine geometry - see Section 3.1. The maximal (proper) flats are the hyperplanes of the matroid, and hence the cocircuits are the set theoretic complements of these. The hyperplanes of M correspond to sets of collinear points.

An example follows to help visualize the different sets:

Example: The illustration in Figure 5 is a rank 3 subspace of $\mathbb{Z}^2$. The following lists give the elements of this subspace which are independent, dependent, circuits etc.

The cocircuits of M form the circuits of a dual matroid, M', and we partition the circuits into positive and negative components depending on which side of the defining hyperplane they fall. (Note that this step is not generally feasible, as it assumes the underlying geometry of the discrete space.) This partitioning of the circuits of M' gives us the oriented matroid OM.



Independents:	∅, {a}, {b}, {c}, {d}, {e}, {a,b}, {a,c}, {a,d}, {a,e}, {b,c}, {b,d}, {b,e}, {c,d}, {c,e}, {d,e}, {a,b,c}, {a,b,d}, {a,b,e}, {a,c,d}, {a,c,e}, {b,c,e}, {c,d,e}
Bases:	{a,b,c}, {a,b,d}, {a,b,e}, {a,c,d}, {a,c,e}, {b,c,e}, {c,d,e}
Dependents:	{a,d,e}, {b,c,d} and all sets of 4 or more points.
Circuits:	{a,d,e}, {b,c,d}, {a,b,c,e}
Subspaces:	(rank 0) ∅ , (rank 1) all singleton sets, (rank 2) {a,b}, {a,c}, {b,e}, {c,e}, {a,d,e}, {b,c,d}
Hyperplanes:	{a,b}, {a,c}, {b,e}, {c,e}, {a,d,e}, {b,c,d}
Co-circuits:	{c,d,e}, {b,d,e}, {a,c,d}, {a,b,d}, {b,c}, {a,e}

Fig. 5. A subset of $\mathbb{Z}^2$

Example: The following pairs give the signed circuits of an oriented matroid on the points in Figure 5: ({c},{d,e}), ({b,d,e},∅), ({c},{a,d}), ({a,b,d},∅), ({b,c},∅), ({a},{e}) and their opposite pairs.

We define halfspaces to be the complement of the positive elements of a circuit. This gives us the means to generate a convex geometry on the oriented matroid. We define a *convex subset* of an oriented matroid as being any subset of its ground set equal to the intersection of halfspaces of the matroid. This set of convex subsets indeed satisfies the axioms for a convex geometry:

Theorem 1. *Let $X \subseteq \mathbb{Z}^2$ be a finite set, and $M = (X, (C^+, C^-))$ be an oriented matroid defined in the manner described above. Then, the set of convex subsets of M, $\mathcal{C}$, is a convex geometry on X.*

Proof.

A0 Take any maximal independent set. Then, take the set of hyperplanes containing the maximal proper subsets of this set. Now take the halfspaces such that the signed circuit part of the halfspace does not contain any elements of the independent set. Then the intersection of these halfspaces is empty.

A1 Since the space is finite, it is therefore bounded. Take any hyperplane such that the remainder of the space is entirely to one side, then we have a halfspace which is in fact the entire space.

A2 Since all convex sets are defined as an intersection of halfspaces, the intersection of sets of convex sets are by definition intersections of sets of halfspaces, and hence are convex sets themselves.

A3 Given a set of convex sets totally ordered by inclusion, the largest of these will be their union, and is of course a convex set.

AE (Idea of proof) We need to show that for two points $x, y \notin [S]$ that if $y \in [S \cup \{x\}]$ then $x \notin [S \cup \{y\}]$. We can show that x is an extreme point ('outside vertex') of $S \cup \{x\}$. Since $y \in [S \cup \{x\}]$, then $[S \cup \{x\}] = [S \cup \{x\} \cup \{y\}]$ and so x is an extreme point of $S \cup \{x\} \cup \{y\}$. We can show that a hyperplane exists separating an extreme point from the rest of the set, and hence a halfspace exists containing $S \cup \{y\}$ but not x. Hence $x \notin [S \cup \{y\}]$.

This shows that the oriented matroid structure is rich enough to give us both convex and affine geometries, and demonstrates how these concepts are closely related. Note that in fact due to the way in which the proof works, the theorem holds more generally for finite subsets of $\mathbb{R}^2$.

4.2 Convexity Operators on Cell Complexes in $\mathbb{Z}^2$

We model a finite subset C of $\mathbb{Z}^2$ here with a 2 dimensional cartesian cell complex. This complex takes singleton sets of points of C as its 0-cells ($dim(\{(n, m)\} = 0)$, sets of points of the form $\{(n, m), (n, m+1)\}$ or $\{(n, m), (n+1, m)\}$ as 1-cells, and sets $\{(n, m), (n+1, m), (n, m+1), (n+1, m+1)\}$ as 2-cells. The bounding relation bnd is simply the subset relation between cells - a bounds b iff $a \subset b$. The complex will be denoted $\mathbb{C} = (C, bnd, dim)$.

We define an oriented matroid on the cell complex in the way described in Section 4.1, where the ground set is the set of 0-cells. Given a halfspace H of the matroid, we call the set of cells $\{c \in C : \ c \subseteq \bigcup H\}$ a *cc-halfspace*. It corresponds to all of the cells either on, or to one side of the hyperplane defining the halfspace. Note that a cc-complex will always be a *closed subcomplex*, in the sense that if a cell is in the subcomplex, then so will all of its bounding cells. Dually, if a cell is in an *open subcomplex* then all of the cells that it bounds will be in the subcomplex. Correspondingly, we define a *oc-halfspace* as the smallest open subcomplex containing a cc-halfspace.

This definition of cc- and oc-halfspaces leads us to two distinct notions of convex hull, closed and open convex hulls:

Closed Convex Hull: The closed convex hull of a set S is the intersection of all cc-halfspaces containing S.

Open Convex Hull: The open convex hull of a set S is the intersection of all oc-halfspaces containing S.

These definitions both apply the notion of convexity as intersecting halfplanes directly to cell complexes. These definitions would be particularly applicable if we wish to restrict the subcomplexes we are interested in to the closed (open) subcomplexes, the choice between the two may depend on whether we take a point or pixel view of the world.

We should note here that though these convexity operators do indeed satisfy the axioms A1-A3 (and the closed convexity operator also satisfies A0), neither satisfies the anti-exchange axiom AE. The first diagram in Figure 6 shows a set S (and its closed convex hull), and labelled cells x and y. The second diagram shows the hull of $S \cup x$ - note that this is the same as the hull of $S \cup y$. This shows that y is in the hull of $S \cup x$, but also x is in the hull of $S \cup y$, and so the anti-exchange axiom fails.

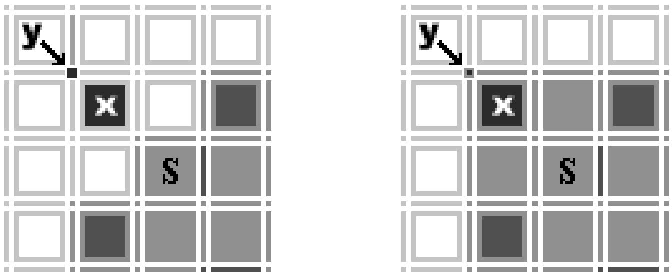


Fig. 6. Closed convex hulls do not satisfy the anti-exchange axiom

Centred Convex Hull: The centred convex hull of a set S is the convex hull of S when the cell complex is considered as an oriented matroid whose ground set is the set of cells, treating cells as their centre points for the purposes of generating the matroid.

This notion simply treats the cell complex as a grid of points twice the size of the set of 0-cells, and as such is clearly a convex geometry. Note that this notion of convexity treats cells of different dimension as if they were all 0-dimension.

Embedded Convex Hull: The embedded convex hull of a set S is the image in the cell complex of the convex hull of S when considered as embedded in Euclidean space.

This is a convenient way of defining a convexity operation on a cell complex, as it applies well known operations in Euclidean space to the cells of a cell complex. However, it is a matter of interpretation as to how the image of a set in Euclidean space in a cell complex is defined. If we define it as the set of cells which overlap the (Euclidean) hull (see [Web01] for example), then we have a hull which is not idempotent.

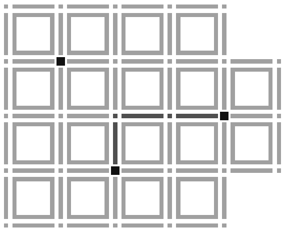


Fig. 7. Closed Convex Hull

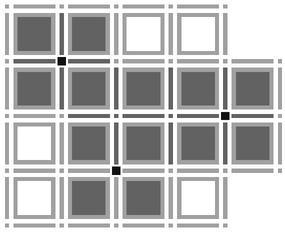


Fig. 8. Open Convex Hull

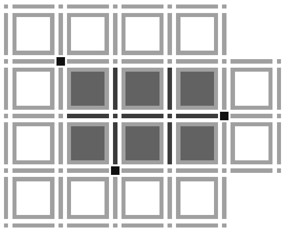


Fig. 9. Embedded Convex Hull

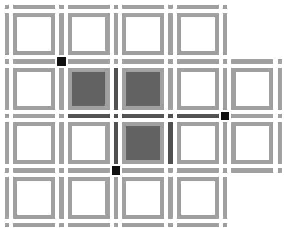


Fig. 10. Centred Convex Hull

Since we are working in a discrete space, we would expect that some aspects of a more traditional continuous notion of convexity will be lost. Indeed the notions of convexity we have investigated can all produce results that may seem at odds with convexity in a Euclidean context, even if they do form a convex geometry. For example there is no guarantee that set which is convex under the closed or centred definitions are connected, embedded convexity is not idempotent, and closed and open convexity do not satisfy the anti-exchange axiom.

Each operator has its uses however, the closed (open) hull is particularly useful in a context where the only sets of interest are closed (open) (see for example [RS02]). The centred convexity operator would be suited to any situation where a robust method of finding the convex hull is desired, and where the hull must satisfy the axioms of convexity. The embedded convexity operator would be most suited to applications where convexity operations are already defined for Euclidean space, and a simple mapping from this to cell complexes is desired.

5 Convexity Algorithms for Oriented Matroids

This section looks at data structures and algorithms using oriented matroids.

5.1 Matroid Data Structure

The basic structure of a Matroid is a set and a set of subsets of that set. The set data structure can be efficiently implemented as a hashtable which is restricted to ignore duplicated items (see for example the Java HashSet class). Such a structure will have constant complexity for checking the *member* relation between an element and a set, and linear complexity for checking the *subset* relation between sets. The structure should be immutable, as there should be no reason to alter the sets once they are initially stored.

The validateIndependents algorithm validates a Matroid under the independence axioms. Axiom I2 states that every subset of an independent set is also independent. We need only check the maximal proper subsets of each independent set however, since due to the transitivity of the subset relation and due to the fact that if a maximal proper subset is independent we will also be checking *its* maximal proper subsets. The following algorithm returns the set of maximal proper subsets of a set (complexity $O(n)$):

Algorithm maximalSubsets(s)

Input: `A set` *s*.

Output: `A set` *out* `of the maximal subsets of s.`

```
create new set out;
for (each e of s)
  add s-{e} to out
end for;
return out;
```

We now give an algorithm which determines whether or not the exchange axiom holds for a pair of sets s_1, s_2 (complexity $O(mn)$ where m and n are the cardinality of each input set):

Algorithm exchange(s1,s2)

Input: `Two sets s1 and s2`

Output: `true if the exchange axiom holds for these sets, false otherwise.`

```
exchanged := false;
diff := s2 - s1;
for (each e of diff)
  temp := s1 ∪ {e};
  exchanged := exchanged OR temp is in independents;
  if (exchanged = true ) return true ;
end for
return false ;
```

The following algorithm validates the independent sets of a matroid (complexity $O(n^2 m)$ where n is the number of independent sets, and m is the cardinality of the ground set):

Algorithm validateIndependents(M)

Input: A matroid *M=(groundset, indeps)*

Output: true if the validation is successful, false otherwise.

```
allSubsets := true ; exchange := true ;
for (each i in indeps)
  if (i is not a subset of groundset) return false ;
  allSubsets := allSubsets AND maximalSubsets(i) ⊆ indeps;
  if (allSubsets = false ) return false ;
  for (each i2 in indeps)
    if (|i| < |i2|) exchange := exchange AND exchange(i, i2);
    if (exchange = false ) return false ;
  end for;
end for;
```

Note that though the complexity is quite high on these operations, we only need to perform validation once, when the matroid is first created. In fact, if we have mathematical proof that a particular set and set of subsets satisfies the axioms, validation will be unnecessary.

5.2 Operations on Oriented Matroids

Since the axioms for an oriented matroid are different from (and not equivalent to) the axioms of matroids, the validation algorithm is different. We denote the opposite of a signed circuit $c = (c^+, c^-)$ by $-c = (c^-, c^+)$. The complexity is $O(m^3 n^2)$ for a oriented matroid with ground set of cardinality n and m circuits. Again, this is a one-time check, and unnecessary in the presence of a proof that a structure satisfies the axioms.

Algorithm validateCircuits(M)

Input: An oriented matroid *M=(groundset, circuits)*

Output: true if the validation is successful, false otherwise.

```
for (each c in circuits)
  if (c is not a subset of groundset) return false ;
  if (c has no elements) return false ;
  if (c ≠ -c) return false;
  for (each c2 in circuits)
    if (c2 ⊆ c AND c2 ≠ c) return false ;
    if (c and c2 are equal) skip to next iteration of loop;
    setInter := intersection(c+, c2-);
    setUnionPos := union(c+, c2+);
```

```
    setUnionNeg := union(c⁻, c2⁻);
    inter : for (each a in setInter)
     testPos := setUnionPos - a;
     testNeg := setUnionNeg - a;
     for (each c3 in circuits)
      hasCirc := c3⁺ ⊆ testPos AND c3⁻ ⊆ testNeg;
      if (hasCirc = true) skip to next iteration of inter;
     return false;
     end for;
    end for;
  end for;
```

The following algorithm generates the set of halfspaces of an oriented matroid. The terminology used is based on the notion that the circuits of the matroid we have in mind are generated by the cocircuits of a matroid whose independent sets are linearly independent in the Euclidean sense. Of course the algorithm will work on any oriented matroid.

Algorithm halfspaces(M)

Input: An Oriented matroid *M=(groundset, circuits)*

Output: A set of subsets of the ground set of *M* , the *halfspaces*.

```
create new set halfspaces
for (each c in circuits )
  create new set subset
  subset = groundset - getPositiveCircuit(c )
  add subset to halfspaces
return halfspaces
```

The function *getPositiveCircuit(c)* simply returns the positive elements of the circuit. The algorithm then simply has the effect of collecting together the set theoretic complements of these half-circuits. The convex hull algorithm works directly with the set of halfspaces of an oriented matroid:

Algorithm convexHull(M,points)

Input: An Oriented matroid, M=(groundset, signed_circuits), and a subset of points of groundset.

Output: A subset of groundset corresponding to the convex hull of points in M.

```
out := groundset;
hspaces := halfspaces(M);
for (each h in hspaces)
  if points ⊆ h then out := intersection(out,h);
end for ;
```

Clearly the convex hull operator works with the Oriented Matroid structure to give a simple and robust algorithm for convex hulls. Once the matroid is instantiated, the algorithm for generating the convex hull is quite efficient, involving just one loop. If the number of elements in the ground set is n, and the number of circuits is m, then the complexity of the hull operator will be $O(nm)$.

6 Conclusions and Further Work

We have shown that it is possible to axiomatize convexity in a discrete setting. The nature of the axioms make oriented matroids a natural structure for capturing much of the geometry required in such a setting, and we have given a construction of oriented matroids from subspaces of $\mathbb{Z}^n$.

The oriented matroid structure can be superimposed on cell complexes, and convexity operations deriving from this underlying matroid ensure that the axioms of convexity are satisfied by particular convexity operations. We have also briefly investigated the properties of different convexity operations, and considered the sort of setting that each might be appropriate in.

Algorithms for obtaining a convex hull through the oriented matroid structure are shown, along with algorithms for determining that a set of subsets are indeed the independent sets, or signed circuits of a (oriented) matroid if mathematical proof is not available.

There is plenty of scope for further development to this work. The algorithms given are not very efficient at present. However we are working on improving these, using techniques similar to those used in convexity algorithms in Euclidean geometry by considering notions of extreme points. We also aim to provide algorithms that generate halfspaces 'on the fly', rather than the memory and time intensive method we currently have in generating all of the halfspaces. We also intend to look further at convexity on cell complexes, with the intent of applying convexity to qualitative spatial reasoning in discrete space.

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