M. Szczuka et al. (Eds): RSCTC 2010, LNAI 6086, pp267-276
Springer-Verlag (2010)

# Relational Granularity for Hypergraphs

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**Abstract.** A set of subsets of a set may be seen as granules that allow arbitrary subsets to be approximated in terms of these granules. In the simplest case of rough set theory, the set of granules is required to partition the underlying set, but granulations based on relations more general than equivalence relations are well-known within rough set theory. The operations of dilation and erosion from mathematical morphology, together with their converse forms, can be used to organize different techniques of granular approximation for subsets of a set with respect to an arbitrary relation. The extension of this approach to granulations of sets with structure is examined here for the case of hypergraphs. A novel notion of relation on a hypergraph is presented, and the application of these relations to a theory of granularity for hypergraphs is discussed.

## 1 Introduction

The theory of rough sets [Paw82,PS07] provides, in its most basic form, a way of approximating arbitrary subsets of a fixed universal set $U$ in terms of the equivalence classes of an equivalence relation on $U$. These equivalence classes can be thought of as granules which represent a coarser, or less detailed, view of $U$ in which we cannot detect individual elements – all we can see are the granules. This initial starting point of the theory has been extended, [SS96,Lin98], to more general relations on $U$, including the case of an arbitrary binary relation [Yao98,Zhu07]. For a relation $R$ on $U$, the granules are the neighbourhoods, that is subsets of the form $R(x) = \{y \in U \mid x \; R \; y\}$. More generally still, a binary relation between two sets can be used, and the numerous links that this reveals between rough sets and other topics including formal concept analysis, Chu spaces, modal logic, and formal topology are discussed in detail by Pagliani and Chakraborty [PC08] in their monograph on rough set theory. The general importance of granularity in information processing has been discussed by numerous authors including [Zad97,Yao01].

Rough set theory represents a substantial body of knowledge which encompasses both practical techniques for data analysis and theoretical results in logic and algebra. As its name indicates, the fundamental concern of the theory is with sets, that is with collections of entities having no additional structure. However,

* Supported by EPSRC (EP/F036019/1) and Ordnance Survey project *Ontological Granularity for Dynamic Geo-Networks*

granularity presents significant challenges in other contexts, and the focus in this paper is on the case that we do not merely have a set of entities where we need to approximate subsets, but where we have a graph (or more generally a hypergraph) and we need to give approximate descriptions of subgraphs. We can regard this as an extension of rough set theory since a set is just a special kind of graph in which there are only nodes (or vertices) and no edges (or arcs). For a simple example we can consider a graph as modelling a railway network and we can imagine a process of granulation in which we take a less-detailed view of the network. The need for such a granulation might arise from incomplete knowledge of some event affecting the network, such as an accident or a terrorist incident. It might also arise from the particular needs of users: a passenger requires a view of the network which is different from that of an engineer working to maintain part of the system. Hypergraphs generalize graphs by allowing edges that may be incident with arbitrary sets of edges rather than with just one or two edges. In giving a granular view of a railway network we might need to indicate that it is possible to travel between any two stations in some set without specifying exactly what station-to-station links exist. This kind of scenario is one reason why it is appropriate to consider hypergraphs and not just graphs.

In order to understand what rough hypergraph theory might be, this paper considers how a relation on a hypergraph can be used to define approximation operators which generalize the operators defined for a relation on a set by Yao [Yao98]. It turns out that the approximation operators defined by Yao can be related to well-known constructions in mathematical morphology, and this can be used as a way of generalizing the operators to ones on hypergraphs. Mathematical morphology [Ser82] originated in image processing but the most basic aspects of the body of techniques it provides (erosions, dilations, openings and closings) can all be presented [BHR07] in terms of binary relations. Although connections between rough sets and mathematical morphology have studied [Blo00,Ste07], there appears to be potential for this topic to contribute further to a general understanding of granularity.

It is not immediately clear what we should mean by a relation on a hypergraph. One possibility would be two separate relations, one for edges and one for nodes, subject to some compatibility condition. The disadvantage of adopting this approach is that we find such relations do not correspond to the sup-preserving operations on the lattice of sub-hypergraphs. This is significant, because the well-known fact that relations on a set, $U$, are equivalent to sup-preserving operations on the powerset $\mathscr{P}U$ is an essential ingredient in mathematical morphology. If we are to take advantage of the way mathematical morphology provides operations for granular approximation, we need the appropriate definition of hypergraph relations.

Section 2 describes how existing rough set approximation operators can be described using morphological operators. Hypergraphs are introduced formally in Section 3, where it emerges that the hypergraph relations we need must allow edges to be related to nodes as well as to edges, and dually nodes may be related to edges as well as to nodes. The main technical challenge solved by the paper concerns the converse of a hypergraph relation. Although our relations can be modelled as sets of arrows, simply reversing the direction of the arrows fails to give a valid relation. Section 4 shows how to construct the converse and shows that it performs the same role with respect to sub-hypergraphs as the converse of a usual relation does with respect to subsets. Limitations of space mean that proofs have been omitted, but it is hoped that that the inclusion of examples of the approximation operators obtained in Section 5 will allow the reader to appreciate the main features of the ideas introduced. Conclusions and further work appear in Section 6.

## 2 Approximation Operators

The purpose of this section is to recall the six approximation operators described in [Yao98] and to relate them to operators from mathematical morphology. This will be used later as the means of seeing how to generalize these operators when we have a relation on a hypergraph rather than a relation on a set.

Suppose we have a set $U$ and a subset $A \subseteq U$. To give a granular, or less detailed, description of $A$ is to describe $A$ not in terms of the elements it contains, but in terms of certain subsets of $U$ called granules. These granules can be thought of as arising from some notion of indistinguishability on the elements of $U$. From this viewpoint, a granule clumps together elements of $U$ that are not distinguished from each other. Granules often arise from a binary relation on $U$.

**Definition 1** *Let $R$ be a relation on $U$, then the* ***granules*** *(with respect to $R$) are the subsets $R(x) = \{y \in U \mid x \; R \; y\}$ where $x \in U$.*

When there are no restrictions on $R$, an element of $U$ may belong to many granules or none. Given a relation $R$, each subset $A \subseteq U$ can be described in terms of the granules. These arise from two ways in which a set of elements gives rise to a set of granules, and two ways in which a set of granules gives rise to a set of elements. From a set of elements $A \subseteq U$ we can take the granules that intersect $A$, or the granules that are subsets of $A$. From a set of granules $G$ we can take the elements where at least one of their granules is present in $G$, or we can take the elements all of whose granules lie in $G$. These possibilities yield four approximations to $A$, and I use the notation for these used in [Yao98]. If the relation $R$ is not clear from the context, we can write $\underline{apr}'_R(A)$ etc.

$$
\begin{aligned}
\underline{apr}'(A) &= \{x \in U \mid \exists y \in U (x \in R(y) \wedge R(y) \subseteq A)\} \\
\underline{apr}''(A) &= \{x \in U \mid \forall y \in U (x \in R(y) \Rightarrow R(y) \subseteq A)\} \\
\overline{apr}'(A) &= \{x \in U \mid \forall y \in U (x \in R(y) \Rightarrow R(y) \cap A \neq \varnothing)\} \\
\overline{apr}''(A) &= \{x \in U \mid \exists y \in U (x \in R(y) \wedge R(y) \cap A \neq \varnothing)\}
\end{aligned}
$$

In addition to these four operators, there are two further ones [Yao98, p246]:

$$
\begin{aligned}
\underline{apr}(A) &= \{x \in U \mid \forall y \in U (y \in R(x) \Rightarrow y \in A)\}, \\
\overline{apr}(A) &= \{x \in U \mid \exists y \in U (y \in R(x) \wedge y \in A)\}.
\end{aligned}
$$

These six operators can be represented in terms of dilations and erosions as used in mathematical morphology. The relation $R : U \to U$ has an associated function, known as a dilation, $\_ \oplus R : \mathscr{P}U \to \mathscr{P}U$ defined by $A \oplus R = \{x \in U \mid \exists y \in U(y\ R\ x \wedge y \in A)\}$. The dilation is a sup-preserving a mapping between complete lattices, so has a right adjoint $R \ominus \_ : \mathscr{P}U \to \mathscr{P}U$ which can be described by $R \ominus A = \{x \in U \mid \forall y \in U(x\ R\ y \Rightarrow y \in A)\}$.

It is necessary here to assume some knowledge of adjunctions on posets (or Galois connections), but details can be found in [Tay99]. The notation $f \dashv g$ will be used when $f$ is left adjoint to $g$, and the idea [Tay99, p152], of viewing $\dashv$ as an arrow (with the horizontal dash as the shaft of the arrow, and the vertical dash as the head of the arrow) proceeding from the left adjoint to the right adjoint is also adopted. This leads to diagrams of the form $\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}$ and various rotated forms in Section 4 below.

Although writing dilations on the right and erosions on the left is contrary to established practice in mathematical morphology, it is adopted here since if we have a second relation $S$ then using ; to denote composition in the 'diagrammatic' order $(R\,;S) \ominus A = R \ominus (S \ominus A)$ and $A \oplus (R\,;S) = (A \oplus R) \oplus S$. The relation $R$ has a converse $R^{-1}$ and dilation and erosion by $R^{-1}$ yield operations (note the sides on which these act) $R \oplus^{-1} \_ : \mathscr{P}U \to \mathscr{P}U$ and $\_ \ominus^{-1} R : \mathscr{P}U \to \mathscr{P}U$.

The following result is stated without proof, as it follows by routine techniques. The approximations $\underline{apr}'$ and $\overline{apr}''$ are particularly well-known in mathematical morphology as the opening and as the closing by the converse.

**Theorem 1** *For any relation $R$ on $U$ and any $A \subseteq U$,*

$$\begin{aligned} \underline{apr}(A) &= R \ominus A, & \overline{apr}(A) &= R \oplus^{-1} A, \\ \underline{apr}'(A) &= (R \ominus A) \oplus R, & \overline{apr}'(A) &= (R \oplus^{-1} A) \ominus^{-1} R, \\ \underline{apr}''(A) &= (R \ominus A) \ominus^{-1} R, & \overline{apr}''(A) &= (R \oplus^{-1} A) \oplus R. \end{aligned}$$

## 3 Relations on Hypergraphs

A hypergraph [Ber89] is a generalization of the concept of undirected graph in which an edge (or rather a 'hyperedge') may be incident with more than two nodes. As with graphs there are many variants of the basic idea. In the present work hypergraphs are permitted to have two distinct hyperedges incident with the same set of nodes, and hyperedges incident with an empty set of nodes are also allowed. Formally, a hypergraph consists of two sets $E$ (the hyperedges) and $N$ (the nodes) together with an arbitrary function from $E$ to the powerset $\mathscr{P}N$.

A hypergraph may be drawn as in Figure 1 with each hyperedge as a closed curve containing the nodes with which it is incident. The example includes a hyperedge, $f$, incident with no nodes, and a node, $z$, to which no hyperedges are incident. The hyperedge $e$ is incident with exactly one node, a situation that would correspond to a loop on the node $y$ in an ordinary graph.

The idea of a hypergraph as having two disjoint sets of hyperedges and nodes is useful, but it turns out to be not the most appropriate for our purposes. Instead

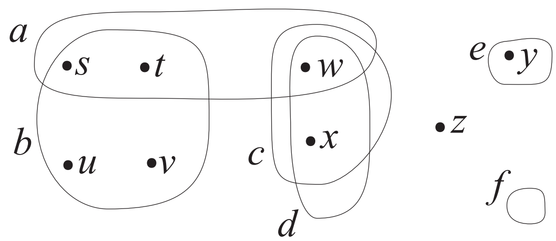



**Fig. 1.** Example of a hypergraph with hyperedges $\{a, b, c, d, e, f\}$ and nodes $\{s, t, u, v, w, x, y, z\}$

we need a definition based on the approach to graphs found in [BMSW06] and used in [Ste07]. This means we have a single set of elements comprising both edges and nodes and a relation associating nodes to themselves and edges to their incident edges.

**Definition 2** *A **hypergraph** $(H, \varphi)$ is a set $H$ together with a relation $\varphi : H \to H$ such that for all $x, y, z \in H$ if $x \varphi y$ then $y \varphi z$ if and only if $y = z$.*

Figure 2 shows the same hypergraph as in Figure 1 visualized as a binary relation.

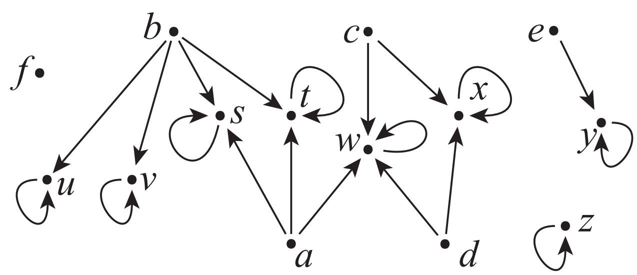



**Fig. 2.** The hypergraph in Figure 1 as a binary relation

Moving to consider relations on hypergraphs, we start with the definition of sub-hypergraph, which is essentially the requirement that whenever a hyperedge is present then all nodes with which it is incident are present too.

**Definition 3** *A **sub-hypergraph** of $(H, \varphi)$ is a subset $K \subseteq H$ such that $K \oplus \varphi \subseteq K$.*

It may be checked that the sub-hypergraphs form a complete lattice which is a sub-lattice of the powerset $\mathscr{P}H$. This lattice of sub-hypergraphs will be denoted

$\mathscr{L}\varphi$. The inclusion of $\mathscr{L}\varphi$ in $\mathscr{P}H$ has both left and right adjoints constructed as in the following definition.

**Definition 4** *Let $A \subseteq H$, then we define the two ways to make $A$ into a hypergraph* $\uparrow A = \bigcap\{K \in \mathscr{L}\varphi \mid A \subseteq K\}$ *and* $\downarrow A = \bigcup\{K \in \mathscr{L}\varphi \mid K \subseteq A\}$.

Given a hypergraph $(H, \varphi)$, let $I_\varphi$ be the relation $I_H \cup \varphi$ on the set $H$, where $I_H$ denotes the identity relation on the set $H$.

**Definition 5** *A* ***hypergraph relation*** *on $(H, \varphi)$ is a relation $R$ on the set $H$ for which $R = I_\varphi \,;\, R \,;\, I_\varphi$.*

These relations play the same role with respect to the lattice $\mathscr{L}\varphi$ as the ordinary relations on the set $H$ do with respect to the lattice $\mathscr{P}H$.

**Theorem 2** *The hypergraph relations on $(H, \varphi)$ form a quantale [Ros90] under composition of relations (with unit $I_\varphi$) which is isomorphic to the quantale of sup-preserving mappings on $\mathscr{L}\varphi$.*

An example of a hypergraph relation is shown in figure 3. In this example the hypergraph is actually a graph. The relation is shown by the dashed lines.


Graph $G$ with edges $\{a, b, c\}$ and nodes $\{x, y, z\}$.

Relation $R$ on $G$. $R = \{(a, a), (a, x), (a, z), (b, x), (c, a), (c, x), (c, z), (x, x), (y, x)\}$

**Fig. 3.** A Relation on a graph with three nodes and three edges

## 4 Converse for Hypergraph Relations

When $R$ is a hypergraph relation, $R^{-1}$ (the converse in the usual sense) need not be a hypergraph relation. Converse relations appear in the the approximation operators described in Theorem 1 and the notion of equivalence relation depends on symmetry and thus on the converse. To generalize these concepts to hypergraph relations requires that we can construct converses.

First recall one way of characterizing the converse of a relation on a set. Consider the set relation $R$ as sup-preserving mapping $R : \mathscr{P}H \to \mathscr{P}H$ with right adjoint $\Sigma$. The converse can be obtained by defining $R^{-1}(A) = -(\Sigma(-A))$

where $-$ is the set-theoretic complement. This situation is summarised in the diagram on the left of Figure 4. In the diagram the powerset $\mathscr{P}H$ is distinguished from from its opposite, $(\mathscr{P}H)^{\mathrm{op}}$ which has the same elements but with the reversed partial order. The mapping $(R^{-1})^{\mathrm{op}}$ has the identical effect on elements as $R^{-1}$ but the distinction is important for the adjoints.

To generalize the notion of converse to hypergraph relations we replace the complement operation in $\mathscr{P}H$ by the corresponding construction for $\mathscr{L}\varphi$. The lattice $\mathscr{L}\varphi$ is not in general complemented, but there are two weaker operations.

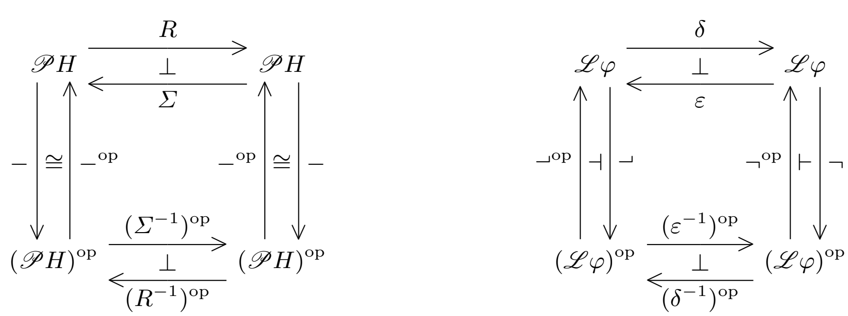
**Definition 6** *Let $K \in \mathscr{L}\varphi$. Then the pseudocomplement $\neg K$ and the dual pseudocomplement $\invneg K$ are given by $\neg K = \downarrow(-K)$ and $\invneg K = \uparrow(-K)$.*

The complement operation $-$ provides an isomorphism between $\mathscr{P}H$ and its opposite. The pseudocomplement and its dual are not in general isomorphisms, but they do satisfy the weaker property of being adjoint to their opposites. That is, for $\neg, \invneg : \mathscr{L}\varphi \to (\mathscr{L}\varphi)^{\mathrm{op}}$, we have $\neg \dashv \neg^{\mathrm{op}}$ and $\invneg^{\mathrm{op}} \dashv \invneg$.

We now come to the definition of the converse of a hypergraph relation. For a relation $R$ the notation $R^c$ is used since $R$ will also have a distinct converse, i.e. $R^{-1}$, as a relation on the set $H$.

**Definition 7** *Let $R$ be a hypergraph relation on $(H, \varphi)$ and $\delta : \mathscr{L}\varphi \to \mathscr{L}\varphi$ its corresponding dilation. Then the converse of $R$ is the hypergraph relation $R^c$ corresponding to $\delta^c : \mathscr{L}\varphi \to \mathscr{L}\varphi$ where $\delta^c(K) = \invneg\varepsilon\neg(K)$ and $\delta \dashv \varepsilon$.*

The situation is summarised in the diagram on the right of Figure 4. The next



The set case

The hypergraph case

**Fig. 4.** Converse via complementation and adjoints

theorem gives a practical means of computing converses as the composition $I_\varphi ; R^{-1} ; I_\varphi$ is more easily calculated than the expression given in Definition 7.

**Theorem 3** *For any hypergraph relation $R$ with associated dilation $\delta : \mathscr{L}\varphi \to \mathscr{L}\varphi$ the converse dilation satisfies $\delta^c K = I_\varphi(R^{-1}K)$ for every subgraph $K$, and the hypergraph relation representing $\delta^c$ is $I_\varphi ; R^{-1} ; I_\varphi$.*

# 5 Examples

The six approximation operators for subsets summarized in section 2 above can now be generalized to operators on sub-hypergraphs by interpreting the descriptions in Theorem 1 using dilations and erosions on the lattice $\mathscr{L}\varphi$ in place of the powerset lattice, and the construction of Theorem 3 for the converse. Examples of these approximations are given in Figures 5, 6, and 7.

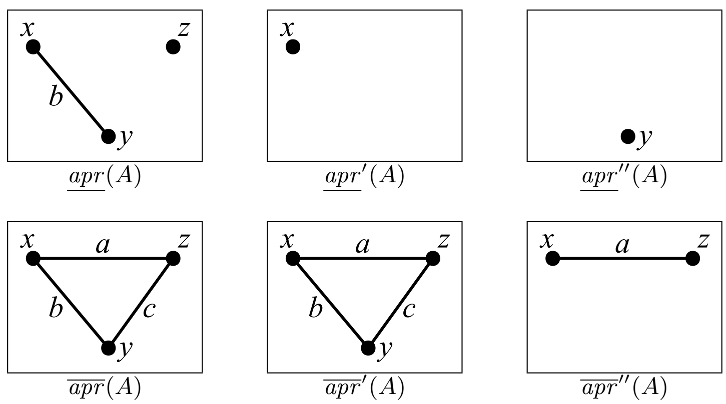
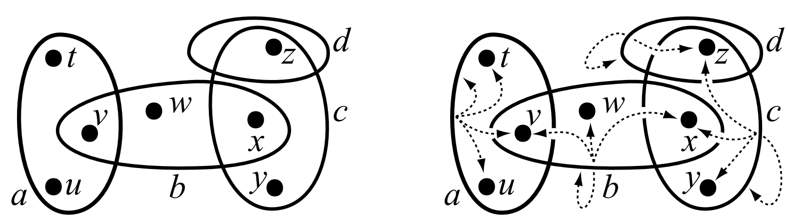



**Fig. 5.** Approximations of the subgraph $A = \{b, c, x, y, z\}$ of $G$ using relation $R$ from Figure 3.




Hypergraph $K$ with edges $\{a, b, c, d\}$ and nodes $\{t, u, v, w, x, y, z\}$

Relation $R = \{(a, a), (a, t), (a, v), (a, u), (b, b), (b, v), (b, w), (b, x), (c, c), (c, x), (c, y), (c, z), (d, d), (d, z)\}$

**Fig. 6.** Second example of a hypergraph relation.

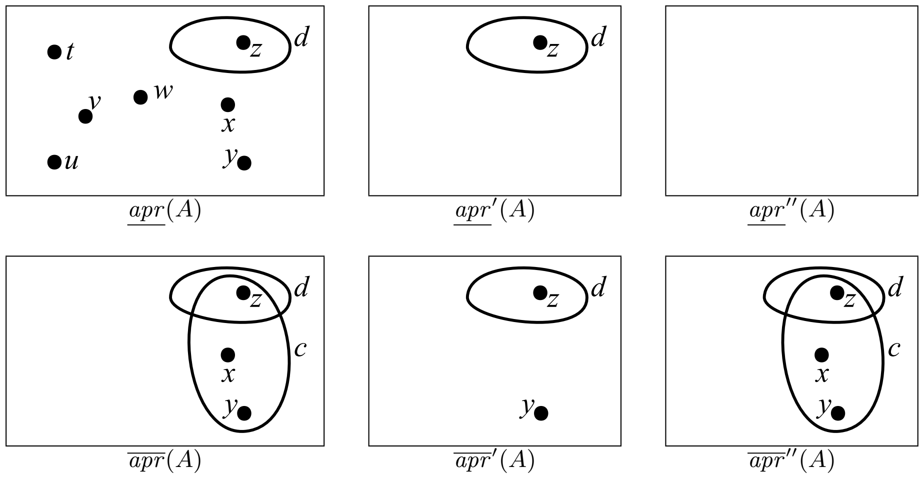



**Fig. 7.** Approximations of the subgraph $A = \{d, z\}$ under the relation $R$ from Figure 6.

## 6 Conclusions and Further Work

This paper has presented a novel approach to granularity for hypergraphs using a view of mathematical morphology as a theory of granularity in order to generalize six approximation operators from sets to hypergraphs. To define these operators on hypergraphs it was necessary to establish appropriate definitions for relations on hypergraphs and for the converse of a relation on a hypergraph. The definition of hypergraph relation has been justified by its equivalence to the notion of sup-preserving mapping on the lattice of sub-hypergraphs. The principal technical achievement in the paper has been the description of the converse of a hypergraph relation.

This work provides a starting point from which it should be possible to develop a full account of rough graphs and hypergraphs which generalizes the existing theory of rough sets. While the six kinds of approximation can all be applied to hypergraphs now that we have established appropriate generalizations of converse dilations and erosions, the properties of these constructions are not necessarily the same as in the set case. The study of these constructions thus presents one direction for further work. Other areas include extending the analysis using a relation on a single hypergraph to relations between distinct hypergraphs, and an investigation of an analogue of equivalence relations on hypergraphs. This latter issue is not straightforward as the notion of symmetry for hypergraph relations appears to have very weak properties related to the properties of the converse operation – in general $(R^c)^c \neq R$ unlike the familiar $(R^{-1})^{-1} = R$.

**Acknowledgements**

I am grateful to three anonymous reviewers for their helpful comments, and in one particular case for much detailed advice about the existing literature.

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