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Boolean connection algebras: A new approach to the Region-Connection Calculus

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Abstract

The Region-Connection Calculus (RCC) is a well established formal system for qualitative spatial reasoning. It provides an axiomatization of space which takes regions as primitive, rather than as constructions from sets of points. The paper introduces Boolean connection algebras (BCAs), and proves that these structures are equivalent to models of the RCC axioms. BCAs permit a wealth of results from the theory of lattices and Boolean algebras to be applied to RCC. This is demonstrated by two theorems which provide constructions for BCAs from suitable distributive lattices. It is already well known that regular connected topological spaces yield models of RCC, but the theorems in this paper substantially generalize this result. Additionally, the lattice theoretic techniques used provide the first proof of this result which does not depend on the existence of points in regions. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Formal treatments of space generally take one of two starting points. It is possible, on the one hand, to take points as the primitives, and build regions out of sets of points. This is the conventional approach adopted in point-set topology [36,43]. On the other hand, theories of space can be constructed in which regions are taken as primitives, and points, if they be admitted at all, are constructed as limiting cases of regions in some way.

Within AI region-based models of space have been proposed within what is known as 'qualitative spatial reasoning' (QSR). The field of QSR can be seen as an area within that

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part of AI which seeks to provide an account of everyday, or commonsense, reasoning about the physical world. This area is known as qualitative reasoning, and its accounts are contrasted with the essentially quantitative ones provided by conventional physics. Qualitative reasoning includes among its concerns everyday tasks carried out by humans such as pouring a liquid from one container to another. Humans have considerable expertise in performing such tasks, despite usually having no appreciation of the mathematical intricacies of theoretical hydrodynamics. Thus qualitative reasoning aims to model such tasks in a way which is much closer to the conceptual models apparently used by humans than the models provided by conventional applied mathematics and physics. Many everyday tasks, such as rearranging furniture or papers on a desktop, involve some appreciation of the space within which the objects exist. It is the specifically spatial aspects of such tasks of which QSR seeks to provide an account. It has been argued [30] that the usual mathematical models of space, including point-set topology, are ill suited to describing spatial concepts as actually employed by humans. This view that new approaches to modelling space are called for in QSR is an important factor in current interest in region-based spatial formalisms.

Qualitative descriptions of space are not restricted in their application to commonsense tasks such as the above examples. They are, for example, relevant to Geographic Information Systems (GIS), where qualitative descriptions of how two geographic regions are related to each other have been widely studied [7,25]. QSR also impinges on linguistics and psychology, having application to understanding spatial expressions within natural language [24] and wayfinding both in small scale and large scale environments [55].

Models of space which have been proposed as appropriate for tackling the kinds of problems mentioned above have not been exclusively region-based. However, researchers approaching these problems from the viewpoint of AI or philosophy have often found region-based spatial formalisms the more natural to work with. The objection to taking points as primitive is often that they have no counterpart in human experience. As Simons puts it [57, p. 42],

> "... no one has ever perceived a point, or will ever do so, whereas people have perceived individuals of finite extent. So the natural *philosophical* approach is to treat points and other boundaries as in some sense ideal abstractions or limits arrived at by approximation from individuals alike in kind with those which are experienced."

The view that lines and higher dimensional spaces cannot be built from points is by no means a recent one. Quotations from several prominent thinkers, dating back to Aristotle, who have taken this view can be found in [3, pp. 1, 2]. Some references to early twentieth century work on point-free accounts of space can be found in [66, p. 73ff and p. 116]. An introduction to pointless geometries is provided by Gerla [27].

Of the early twentieth century researchers, two are especially significant for this paper. Both Whitehead [69] and de Laguna [16] advanced region-based accounts of space in which a relation of 'connection' between regions played an important role. Informally regions are connected if they overlap or if solids occupying them would be in contact. Contemporary with Whitehead and de Laguna was Jean Nicod (1893–1924) whose thesis "Geometry in the Sensible World" appeared in [44]. Whitehead's system was later used as the basis for Clarke's work, published in the 1980s [11,12]. It was this work of Clarke which provided the starting point for the Region-Connection Calculus (RCC) which is the subject of the present paper. RCC was initially described in [52,53], and since then has been developed in an extensive series of papers by Cohn's group at the University of Leeds, and has also been studied by several other researchers. A comprehensive collection of references to papers on RCC can be found in the survey article [13].

RCC consists of a set of axioms which are intended to characterize spatial regions from a qualitative perspective. The axioms constrain two binary operations, sum and product, a unary operation, complement, and a binary relation, connection. Sums and products of regions correspond to unions and intersections of regions. The complement of a region is that region outside it, and two regions are connected if they overlap or touch.

In a recent report [28], Gotts considers the question of what particular structures are models for the RCC axioms. Gotts shows that certain topological spaces, the regular connected ones, provide models of the RCC axioms by taking a region to mean a non-empty regular closed set. Taking non-empty regular open sets also gives a model. These models have the disadvantage that in justifying them, reference is made to points within regions. Gotts observes that "*Using an interpretation expressed in terms of point-sets might seem inconsistent with the spirit underlying the RCC approach. However no alternative has been worked out in any detail, ...*". The detailed proofs provided in [28] are needed in that they justify that certain structures are models of RCC. The disadvantage of the approach taken in [28] is not that the regions have points, but that these points are referred to in the justification. If the only known models of RCC required points to justify them, it would bring into question the extent to which RCC is really a region based formalism—rather than a point-based one in disguise.

The fact that models of RCC are obtained from regular connected topological spaces is established in this paper in a way which makes no reference to points within regions. The results appear as corollaries to Theorems 30 and 39. The idea behind the approach taken to show that points are inessential is based on the fact that the open sets of a topological space form a kind of lattice known as a complete Heyting algebra. This fact has led to one approach to topology in which a space is simply defined to be a complete Heyting algebra. It is thus possible to study topology by using just the lattice theoretic properties of the collection of open sets, without mentioning points at all. This approach is often called "pointless topology", but this does not mean that points are prohibited, only that points are not required. Pointless topology is more general than point-set topology in that there are complete Heyting algebras which do not come from the open set lattices of topological spaces. On the other hand, there are aspects of topological spaces which depend critically on points, and which cannot be described in terms of the open sets alone. Pointless topology has been well developed by mathematicians [34,35,68], but the subject does not seem to have received much attention in the spatial reasoning community. The possibility of using this kind of approach to studying models of RCC was raised by Stell and Worboys [62], but the details presented in the present paper had not been worked out at that stage.

Although the approach taken in this paper to constructing models of RCC is motivated by the basic idea of pointless topology, the constructions are based on structures more general than complete Heyting algebras, and their duals. All the structure which is necessary, can be found in pseudocomplemented distributive lattices and their duals, in particular there is no need to assume that the lattices are complete. This means that the Theorems 30 and 39 are not merely a point-free approach to the results in [28], they are also a substantial generalization of these results. It is a standard result [43, p. 215 ex3] that for a topological space having more than one point and where the one point sets are closed, regularity together with connectedness implies uncountability. Thus models of RCC arising from such spaces have uncountably many points. A specific example of a model of RCC which does not arise as the regular open or regular closed sets of a regular connected topological space appears below in Section 6.

In providing these constructions for models of RCC, it is convenient to provide a reformulation of RCC. This reformulation is based on the concept of a Boolean connection algebra (BCA) which is introduced in Section 3. A BCA, defined formally below, is a pair $\langle A; \mathsf{C}\rangle$, where A is a Boolean algebra, and C is a connection relation on A satisfying four axioms. The concept of BCA is also relevant to another strand in the development of region-based theories of space: mereotopology. Mereology is concerned with parts of entities, and was originally developed by Leśniewski [38,40]. In the spatial context the 'part of' relation between regions is important, and systems of mereotopology have been developed [1,9,58,59,67] which axiomatize spatial regions based on the 'part of' relation among regions and additional topological structure. Mereology has also been used in other areas which are relevant to QSR. For its use in linguistics, see the work of Link [39].

BCAs clarify the mereotopological content of the RCC axioms. In a BCA $\langle A; \mathsf{C}\rangle$ the partial order in A models the notion of part, so the requirement that A be a Boolean algebra is the mereological aspect of the formalism. The topological aspect lies in the axioms for C. The relationship between Boolean algebras and mereology is well known. Tarski [65, p. 333n] notes the close correspondence between complete Boolean algebras and Leśniewski's system of mereology.

The original formulation of RCC, and the earlier work of Whitehead and Clarke, is based on the concept of connection alone. Notions such as part are defined in terms of connection. This use of a single primitive relation is seen by Smith [58, p. 288] as problematic. He objects that deriving the notion of part from that of connection appears contrived, and makes it difficult to separate the mereological and topological aspects of the theory. One reason for Smith's objection to theories which fail to separate mereological and topological aspects, is that modifications to such theories can be difficult to make. This is because changes to one aspect are likely to bring about changes to the other aspect. Boolean connection algebras appear to be conducive to modifications of the kind envisaged by Smith. Some suggestions on how replacing the Boolean algebra, A, by a more general kind of lattice may lead to a theory of vague spatial regions, appear in the final section of this paper.

One of the advantages sometimes claimed for the region-based approach to space is that it is in some sense simpler than approaches based on points. If we start from regions satisfying carefully chosen axioms, it might be expected that some of the more bizarre spatial entities which can be constructed from points, might be excluded from our ontology. However, Pratt and Lemon have demonstrated [50] that this need not be the case. Their work is undoubtedly significant, but it must be observed that region-based systems have not only been motivated by hopes of a simpler ontology. Those quoted in [3, pp. 1–2], mentioned above, appear to take the view that regions should come prior to points in an account of space. This is a view about the nature of space itself, and the proper conceptual basis for a theory of space. It can still be justified even if, in certain circumstances, models of point-based and region-based theories are shown to be equivalent.

There has been some previous work which is related to the lattice theoretic approach taken in this paper. Biacino and Gerla [4] showed that Clarke's system characterized the complete atomless Boolean algebras. They defined a connection structure to be a non-empty set equipped with a binary connection relation and a fusion operator, satisfying four axioms. Associated to each connection structure is an complete orthocomplemented lattice, and when a fifth axiom is satisfied the associated lattice is a complete Boolean algebra. In a later paper [5] they used connection structures in relating definitions of point given by Whitehead and by Grzegorczyk. Boolean algebras as the mereological part of a region-based theory also appear in the work of Roeper [56]. This work, however, makes no mention of Clarke's system or of RCC so further work would be needed to determine the precise relationship between it and BCAs. A forthcoming paper by Düntsch and others represents the work which is most closely related to the present paper. In [22] structures equivalent to BCAs are used and results equivalent to the main ones in Section 4 below are obtained. Düntsch acknowledges my independent derivation of these results. The approach taken by Düntsch is somewhat different from that given below, in that he uses concepts from the theory of relation algebras. Despite this small overlap between [22] and the present paper, detailed proofs are given in Section 4 below, partly in view of the difference in emphasis, but mainly because the material is necessary for that presented in the later sections. Other papers from Düntsch [19,21] have continued to apply techniques from relation algebras.

2. The Region-Connection Calculus

The Region-Connection Calculus was originally formulated [52,53] in a many-sorted logic, LLAMA, due to Cohn. The use of this particular logic was motivated largely by a desire to use the calculus for automated reasoning, thus making efficiency an important consideration. Since the present paper is concerned with certain theoretical issues, and not with implementation, I have not followed the original formulation. The description given below is slightly adapted from that given in [28,30].

2.1. Description of models of RCC

Full details are given below, but the most important ingredients in a model of the Region-Connection Calculus are a set, R, of regions and a binary relation C on R. Various further binary relations on R are defined in terms of C, and these definitions are needed to state the axioms. These include the following, which are intended to capture the ideas of part (P), proper part (PP), overlap (O), external connection (EC), and non-tangential proper part (NTPP).

$\mathsf{P}(x, y)$	iff	For every region z, $\mathsf{C}(z, x)$ implies $\mathsf{C}(z, y)$.
$\mathsf{PP}(x, y)$	iff	$\mathsf{P}(x, y)$ and not $\mathsf{P}(y, x)$.
$\mathsf{O}(x, y)$	iff	There is some region z such that $\mathsf{P}(z, x)$ and $\mathsf{P}(z, y)$.
$\mathsf{EC}(x, y)$	iff	$\mathsf{C}(x, y)$ and not $\mathsf{O}(x, y)$.
$\mathsf{NTPP}(x, y)$	iff	$\mathsf{PP}(x, y)$ and no region z satisfies $\mathsf{EC}(z, x)$ and $\mathsf{EC}(z, y)$.

Definition 1. A *model of the Region-Connection Calculus* consists of a set R, an element $\mathsf{u} \in R$, a singleton set $\{\mathsf{n}\}$ disjoint from R, a unary operation $\mathsf{compl}: R - \{\mathsf{u}\} \to R - \{\mathsf{u}\}$, binary operations $\mathsf{sum}: R \times R \to R$, and $\mathsf{prod}: R \times R \to R \cup \{\mathsf{n}\}$, and a binary relation C on R. These data are required to satisfy the following axioms, which make use of the relations derived from C defined above.

R1. $\forall x \in R \cdot \mathsf{C}(x, x)$.

R2. $\forall x, y \in R \cdot \mathsf{C}(x, y)$ implies $\mathsf{C}(y, x)$.

R3. $\forall x \in R \cdot \mathsf{C}(x, \mathsf{u})$.

R4a. $\forall x \in R \cdot \forall y \in R - \{\mathsf{u}\} \cdot \mathsf{C}(x, \mathsf{compl}\, y)$ iff not $\mathsf{NTPP}(x, y)$.

R4b. $\forall x \in R \cdot \forall y \in R - \{\mathsf{u}\} \cdot \mathsf{O}(x, \mathsf{compl}\, y)$ iff not $\mathsf{P}(x, y)$.

R5. $\forall x, y, z \in R \cdot \mathsf{C}(x, \mathsf{sum}(y, z))$ iff $\mathsf{C}(x, y)$ or $\mathsf{C}(x, z)$.

R6. $\forall x, y, z \in R \cdot \mathsf{prod}(y, z) \in R$ implies
$\mathsf{C}(x, \mathsf{prod}(y, z))$ iff $\exists w \in R \cdot \mathsf{P}(w, y)$ and $\mathsf{P}(w, z)$ and $\mathsf{C}(x, w)$.

R7. $\forall x, y \in R \cdot \mathsf{prod}(x, y) \in R$ iff $\mathsf{O}(x, y)$.

R8. $\forall x \in R \cdot \exists y \in R \cdot \mathsf{NTPP}(y, x)$.

This definition departs from the original formulation of RCC which used two disjoint sets REGION and NULL. Since it is impossible to make distinctions between elements of NULL by means of the operations in RCC, there is no loss of generality in assuming that NULL consists of a single element, n. It should be noted that Düntsch et al. have shown [22, Lemma 4.1] that axiom R7 is redundant.

2.2. *Strict and non-strict models of RCC*

The RCC axioms do not imply that the defined ‘part of’ relation P is antisymmetric. That is, it is possible that $\mathsf{P}(x, y)$ and $\mathsf{P}(y, x)$ without $x = y$ being true. It is thus usual to define the relation EQ by $\mathsf{EQ}(x, y)$ iff $\mathsf{P}(x, y)$ and $\mathsf{P}(y, x)$. The relation EQ will be an equivalence relation, but need not be the identity relation on the set of regions. A model of RCC in which EQ is the identity relation will be called a *strict* model. Because of the definition of P in terms of C, the relation $\mathsf{EQ}(x, y)$ holds if and only if for every $r \in R$, $\mathsf{C}(r, x) \Leftrightarrow \mathsf{C}(r, y)$. Thus, strict models can be characterized as those in which two distinct regions can always be distinguished on the basis of the sets of regions to which each is connected. It should be noted that if we allow non-strict models then RCC differs from

Clarke's mereological system [11, p. 206] since his axiom A0.2 implies that the 'part of' relation is antisymmetric.

To show that non-strict models exist, let $\langle R, \{\mathsf{n}\}; \mathsf{u}, \mathsf{sum}, \mathsf{prod}, \mathsf{compl}, \mathsf{C}\rangle$ be any model of RCC. The structure $\langle R_S, \{\mathsf{n}_S\}; \mathsf{u}_S, \mathsf{sum}_S, \mathsf{prod}_S, \mathsf{compl}_S, \mathsf{C}_S\rangle$, which is defined as follows, then provides an example of a non-strict model.

$$R_S = (R \times \{0\}) \cup \big((R - \{\mathsf{u}\}) \times \{1\}\big),$$
$$\mathsf{n}_S = \langle \mathsf{n}, 0\rangle,$$
$$\mathsf{u}_S = \langle \mathsf{u}, 0\rangle,$$
$$\mathsf{sum}_S(\langle r_1, i_1\rangle, \langle r_2, i_2\rangle) = \langle \mathsf{sum}(r_1, r_2), 0\rangle,$$
$$\mathsf{prod}_S(\langle r_1, i_1\rangle, \langle r_2, i_2\rangle) = \langle \mathsf{prod}(r_1, r_2), 0\rangle,$$
$$\mathsf{compl}_S\langle r, i\rangle = \langle \mathsf{compl}\, r, 0\rangle,$$
$$\mathsf{C}_S(\langle r_1, i_1\rangle, \langle r_2, i_2\rangle) = \mathsf{C}(r_1, r_2).$$

The basic idea in this definition is that R_S consists of two disjoint copies of R. One of the copies of R has the universal region, u, removed. This is necessary to ensure that compl_S is defined on $R_S - \{\mathsf{u}_S\}$. If the elements of R are visualized as planar regions, the elements of R_S can be visualized as located in two planes, with each $\langle r, 1\rangle$ vertically above $\langle r, 0\rangle$. It is straightforward to verify that all the RCC axioms are satisfied by the defined structure. The new model is not strict, since for any $r \in R - \{\mathsf{u}\}$, we have that for any $s \in R_S$, $\mathsf{C}_S(s, \langle r, 0\rangle)$ iff $\mathsf{C}_S(s, \langle r, 1\rangle)$.

Although constructions such as the above show that non-strict models do exist, it is not clear how significant such models are in the practical applications of RCC. It has been suggested by Düntsch [22] that an additional axiom ought to be included in RCC to force all models to be strict. However, it is clear that those who have developed RCC have been well aware of the distinction between EQ and $=$ [13, p. 308 note 11].

3. Boolean connection algebras

This section introduces Boolean connection algebras, and shows that given a Boolean connection algebra we can obtain a model of the Region-Connection Calculus. In the following $\not\mathsf{C}$ is used to denote the complement of the relation C, that is $\not\mathsf{C}(x, y)$ holds iff $\mathsf{C}(x, y)$ does not hold.

Definition 2. Let $A = \langle A; \bot, \top, ', \vee, \wedge\rangle$ be a Boolean algebra with more than two elements, let R denote $A - \{\bot\}$, and let R_- denote $R - \{\top\}$. If C is a binary relation on A, then the structure $\langle A; \mathsf{C}\rangle$, is said to be a *Boolean connection algebra* if it satisfies the following axioms.

A1. C is symmetric, and its restriction to R is reflexive.
A2. $\forall x \in R_- \cdot \mathsf{C}(x, x')$.
A3. $\forall x, y, z \in R \cdot \mathsf{C}(x, y \vee z)$ iff ($\mathsf{C}(x, y)$ or $\mathsf{C}(x, z)$).
A4. $\forall x \in R_- \cdot \exists y \in R \cdot \not\mathsf{C}(x, y)$.

The result that Boolean connection algebras model RCC is presented in two stages: Theorems 4 and 5. The proofs of these theorems both make use of the following lemma.

Lemma 3. *In any Boolean connection algebra,* $\langle A; \mathsf{C}\rangle$, *with* $R = A - \{\bot\}$,

(1) $\forall x, y, z \in R \cdot$ *if* $x \leqslant y$ *and* $\mathsf{C}(x, z)$, *then* $\mathsf{C}(y, z)$,

(2) $\forall x \in R \cdot \mathsf{C}(x, \top)$.

Proof.

(1) Suppose that $x \leqslant y$ and that $\mathsf{C}(x, z)$. By symmetry $\mathsf{C}(z, x)$, and thus by A3, $\mathsf{C}(z, x \vee y)$. As $x \vee y = y$ we get $\mathsf{C}(z, y)$ and hence $\mathsf{C}(y, z)$.

(2) For any $x \in R$ we have $\mathsf{C}(x, x)$ by A1. The first part of the lemma then yields $\mathsf{C}(x, \top)$, since $x \leqslant \top$. □

Theorem 4. *Let* $\langle A; \mathsf{C}\rangle$ *be a Boolean connection algebra. Then the relations derived from* C *can be characterized as follows where* x *and* y *are any elements of* R.

1. $\mathsf{P}(x, y)$ *iff* $x \leqslant y$.
2. $\mathsf{PP}(x, y)$ *iff* $x < y$.
3. $\mathsf{O}(x, y)$ *iff* $x \wedge y > \bot$.
4. $\mathsf{EC}(x, y)$ *iff* $\mathsf{C}(x, y)$ *and* $x \wedge y = \bot$.
5. $\mathsf{NTPP}(x, y)$ *iff* $\begin{cases} x < \top & \textit{when } y = \top, \\ \not\mathsf{C}(x, y') & \textit{when } y \neq \top. \end{cases}$

Proof. As before I will use R_- to denote $R - \{\top\}$. For the characterization of P, Lemma 3 part (1) gives that if $x \leqslant y$ then $\forall z \in R \cdot \mathsf{C}(x, z)$ implies $\mathsf{C}(y, z)$. So we need to consider the converse of this. If $x \nleqslant y$ then $x \wedge y' > \bot$ since A is a Boolean algebra. Now, $x \wedge y' \in R_-$ for if $x \wedge y' = \top$, we get $y' = \top$, which is impossible as $y \in R$. Since $x \wedge y' \in R_-$, we also have $(x \wedge y')' \in R_-$, hence we can apply A4 to obtain a z such that $\not\mathsf{C}(z, (x \wedge y')')$, i.e., $\not\mathsf{C}(z, x' \vee y)$. But this implies $\not\mathsf{C}(z, x')$, and $\not\mathsf{C}(z, y)$ by A3. Now $\mathsf{C}(z, x \vee x')$, by Lemma 3 part (2), so by A3 we have shown that if $x \nleqslant y$ then there is some $z \in R$ such that $\mathsf{C}(x, z)$ and $\not\mathsf{C}(y, z)$.

The characterization of PP follows immediately from that of P.

Using the characterization of P, we have that $\mathsf{O}(x, y)$ iff there is some $z \in R$ such that $z \leqslant x$ and $z \leqslant y$. We have to show that the existence of such a z is equivalent to $x \wedge y > \bot$. If $x \wedge y > \bot$, then we can take $z = x \wedge y$. Conversely, if such a z exists, $\bot < z \leqslant x \wedge y$.

The characterization of EC follows immediately from that of O.

For NTPP, consider first the case of $y = \top$. Using the characterizations of EC and of PP, we have $\mathsf{NTPP}(x, \top)$ iff $x < \top$ and there is no $z \in R$ satisfying four conditions, one of which is $z \wedge \top = \bot$. Since no such z exists, we get $\mathsf{NTPP}(x, \top)$ iff $x < \top$.

Now consider the case of $y < \top$. We have to show that $\not\mathsf{C}(x, y')$ is equivalent to $x < y$ and there being no $z \in R$ satisfying all the four conditions $\mathsf{C}(z, x)$, $x \wedge z = \bot$, $\mathsf{C}(z, y)$, and $y \wedge z = \bot$. Suppose that $\not\mathsf{C}(x, y')$, and that $x \nless y$. By A2 we must have $x \neq y$, so $x \nleqslant y$. This implies that $x \wedge y' > \bot$, so $\mathsf{C}(x \wedge y', x \wedge y')$. Two applications of Lemma 3 part (1) gives the contradiction $\mathsf{C}(x, y')$. Thus we have $x < y$. Now if z exists having the stated properties, then $z \leqslant y'$ since $y \wedge z = \bot$. But then $\mathsf{C}(z, x)$ and Lemma 3 part (1) implies that $\mathsf{C}(x, y')$, a contradiction.

Conversely, suppose that $x < y$ and there is no $z \in R$ satisfying all the four conditions. If $\mathsf{C}(x, y')$ we get a contradiction by putting $z = y'$, using the fact that $\mathsf{C}(y, y')$ by A2. □

Theorem 5. *Let $\langle A; \mathsf{C}\rangle$ be a Boolean connection algebra and make the following definitions.*

$$R = A - \{\bot\},$$
$$\mathsf{n} = \bot,$$
$$\mathsf{sum}(x, y) = x \vee y \quad \textit{for all } x, y \in R,$$
$$\mathsf{prod}(x, y) = x \wedge y \quad \textit{for all } x, y \in R,$$
$$\mathsf{compl}\, x = x' \quad \textit{for all } x \in R - \{\top\},$$
$$\mathsf{u} = \top.$$

Then the structure $\langle R, \{\mathsf{n}\}; \mathsf{u}, \mathsf{compl}, \mathsf{sum}, \mathsf{prod}, \mathsf{C}\rangle$, is a model of the Region-Connection Calculus.

Proof. First note that since A has more than two elements we cannot have $\bot = \top$, so $\mathsf{u} \in R$. It is straightforward to check that the defined operations have the required domains and codomains.

Axioms R1, and R2, are covered by A1. The axiom R3 is satisfied by Lemma 3 part (2). Axiom R4a follows from the characterization of NTPP and the restriction $y \neq \top$ in the axiom. Axiom R4b is equivalent, by the characterizations of P and O, to $x \wedge y' > \bot$ iff $x \nleqslant y$ for all $x \in R$ and all $y \in R_-$. But this holds because A is a Boolean algebra, so R4b is satisfied.

The axiom R5 is satisfied by A3. To justify R6 we have to show that, for any x, y, z in R such that, $y \wedge z > \bot$, the condition $\mathsf{C}(x, y \wedge z)$ is equivalent to the existence of some $w \in R$ such that $w \leqslant y$, $w \leqslant z$, and $\mathsf{C}(x, w)$. If $\mathsf{C}(x, y \wedge z)$ and $y \wedge z > \bot$ we can take $w = y \wedge z$, since $y \wedge z \leqslant y$ and $y \wedge z \leqslant z$. Conversely, given $w > \bot$ such that $w \leqslant y$, $w \leqslant z$, we have $w \leqslant y \wedge z$. Thus, by Lemma 3 part (1), $\mathsf{C}(x, w)$ implies $\mathsf{C}(x, y \wedge z)$.

Axiom R7 is immediate from the characterization of O.

Axiom R8 requires that $\forall x \in R \cdot \exists y \in R \cdot \mathsf{NTPP}(y, x)$. When $x \neq \top$ this is satisfied by A4. When $x = \top$, we have $\mathsf{NTPP}(y, \top)$ iff $y < \top$. Since the Boolean algebra has more than two elements, there must exist y such that $\bot < y < \top$. □

4. Obtaining a BCA from a model of RCC

The previous section has demonstrated that every BCA gives rise to a model of RCC in a natural way. In this section we consider the converse problem: can we obtain a BCA from an arbitrary model of RCC? It turns out that for strict models of RCC there is an exact correspondence with BCAs and the two concepts are really identical. The case of non-strict models is not quite so straightforward. It seems that the strict models are by far the most important ones in qualitative spatial reasoning, so this is the case that we concentrate on.

Suppose that $\langle R, \{\mathsf{n}\}; \mathsf{u}, \mathsf{sum}, \mathsf{prod}, \mathsf{compl}, \mathsf{C}\rangle$ is a strict model of RCC. Define binary operations $\vee$ and $\wedge$ on the set $R \cup \{\mathsf{n}\}$ as follows.

$$x \vee y = \begin{cases} \mathsf{sum}(x, y) & \text{if } x, y \in R, \\ x & \text{if } y = \mathsf{n}, \\ y & \text{if } x = \mathsf{n}, \end{cases} \qquad x \wedge y = \begin{cases} \mathsf{prod}(x, y) & \text{if } x, y \in R, \\ \mathsf{n} & \text{if } \mathsf{n} \in \{x, y\}. \end{cases}$$

Also, define the unary operation $'$ on the set $R \cup \{\mathsf{n}\}$ by $x' = \mathsf{compl}\, x$ for $x \in R - \{\mathsf{u}\}$, and by $\mathsf{u}' = \mathsf{n}$, and $\mathsf{n}' = \mathsf{u}$.

The aim of this section is to show that the structure $\langle R \cup \{\mathsf{n}\}; \mathsf{u}, \mathsf{n}, ', \wedge, \vee, \mathsf{C}\rangle$ is a BCA. The first step is to show that $\langle R \cup \{\mathsf{n}\}; \mathsf{u}, \mathsf{n}, \wedge, \vee\rangle$ is a lattice. To do this we need a preliminary lemma which is easily proved from the RCC axioms.

Lemma 6. *For any $x, y \in R$:*

(1) *If $\mathsf{O}(x, y)$ then $\mathsf{P}(w, \mathsf{prod}(x, y))$ iff $\mathsf{P}(w, x)$ and $\mathsf{P}(w, y)$.*
(2) *If $\mathsf{O}(x, y)$ then $\mathsf{P}(\mathsf{prod}(x, y), x)$.*
(3) $\mathsf{P}(x, \mathsf{sum}(x, y))$.

Lemma 7. *The structure $\langle R \cup \{\mathsf{n}\}; \mathsf{u}, \mathsf{n}, \wedge, \vee\rangle$ has the following properties.*

(1) *$\wedge$ and $\vee$ are associative, commutative and idempotent.*
(2) *$x \vee \mathsf{n} = x$ and $x \wedge \mathsf{u} = x$ for all $x \in R \cup \{\mathsf{n}\}$.*
(3) *$x \wedge (x \vee y) = x = x \vee (x \wedge y)$ for all $x, y \in R \cup \{\mathsf{n}\}$.*

Proof. The details are straightforward, but to illustrate the kind of reasoning involved, we will show part of part (3): the equation $x \wedge (x \vee y) = x$.

The cases where one of x and y is n or u follow immediately from the definitions of $\wedge$ and $\vee$. For the other cases, $\mathsf{P}(\mathsf{prod}(x, (\mathsf{sum}(x, y)), x)$ follows from Lemma 6 part (2), and $\mathsf{P}(x, \mathsf{prod}(x, \mathsf{sum}(x, y)))$ from Lemma 6 parts (1) and (3). Hence $\mathsf{EQ}(\mathsf{prod}(x, \mathsf{sum}(x, y)), x)$, and since we are assuming a strict model, the equation $\mathsf{prod}(x, \mathsf{sum}(x, y)) = x$ follows. □

Having established that we have a lattice, the next step is to show that it is distributive. This turns out to be the most complicated part of the argument that we have a BCA. We first need some preliminary lemmas. Proofs of several of these are straightforward calculations from the RCC axioms, and will not be given here.

Lemma 8. *If $\mathsf{P}(x, y)$ and $\mathsf{C}(x, z)$ then $\mathsf{C}(y, z)$.*

Lemma 9. *For any regions x and y, $\mathsf{O}(x, y)$ or $\mathsf{O}(x, \mathsf{compl}\, y)$.*

Lemma 10. *For any region x there is no r such that $\mathsf{P}(r, x)$ and $\mathsf{P}(r, \mathsf{compl}\, x)$.*

Lemma 11. *If regions x and y do not overlap, then $\mathsf{EQ}(x, \mathsf{prod}(x, \mathsf{compl}\, y))$.*

Proof. Since x does not overlap y it overlaps $\mathsf{compl}\, y$ by Lemma 9. Thus Lemma 6 part (2) implies $\mathsf{P}(\mathsf{prod}(x, \mathsf{compl}\, y), x)$. To show $\mathsf{P}(x, \mathsf{prod}(x, \mathsf{compl}\, y))$, use axiom R6 taking $w = x$. □

Lemma 12. *If regions x and y are such that x overlaps both y and the complement of y, then* $\mathsf{EQ}(x, \mathsf{sum}(\mathsf{prod}(x, y), \mathsf{prod}(x, \mathsf{compl}\, y)))$.

Proof. Denote $\mathsf{sum}(\mathsf{prod}(x, y), \mathsf{prod}(x, \mathsf{compl}\, y))$ by $\hat{x}$. First we show that $\mathsf{P}(\hat{x}, x)$. From Lemma 6 we have $\mathsf{P}(\mathsf{prod}(x, y), x)$, and $\mathsf{P}(\mathsf{prod}(x, \mathsf{compl}\, y), x)$. Now if a region r is connected to $\hat{x}$, it must, by R5, be connected to one of $\mathsf{prod}(x, y)$ and $\mathsf{prod}(x, \mathsf{compl}\, y)$. In either case we conclude $\mathsf{C}(r, x)$ by Lemma 8.

Secondly, we show that $\mathsf{P}(x, \hat{x})$. If this is not the case $\mathsf{O}(x, \mathsf{compl}\, \hat{x})$ by R4b. Now $\mathsf{prod}(x, \mathsf{compl}\, \hat{x})$ overlaps at least one of y and $\mathsf{compl}\, y$ by Lemma 9. If it is y, then there exists a region r which is a part of y and also of $\mathsf{prod}(x, \mathsf{compl}\, \hat{x})$. Hence $\mathsf{P}(r, \mathsf{prod}(x, y))$, and $\mathsf{P}(r, \hat{x})$. But we have $\mathsf{P}(r, \mathsf{compl}\, \hat{x})$, which is a contradiction by Lemma 10. The case of $\mathsf{prod}(x, \mathsf{compl}\, \hat{x})$ overlapping $\mathsf{compl}\, y$ is similar. □

Lemma 13. *If* $\mathsf{PP}(x, y)$ *and* $\mathsf{P}(y, z)$ *then* $\mathsf{PP}(x, z)$.

Lemma 14. *If* $\mathsf{NTPP}(x, y)$ *and* $\mathsf{P}(y, z)$ *then* $\mathsf{NTPP}(x, z)$.

Proof. If $\mathsf{NTPP}(x, y)$ and $\mathsf{P}(y, z)$ then x is a proper part of z by Lemma 13. Suppose r is externally connected both to x and to z. Since r is connected to x which is a proper part of y, we get that r is connected to y. But r cannot be externally connected to y as $\mathsf{NTPP}(x, y)$. So r overlaps y and hence z contradicting $\mathsf{EC}(r, z)$. □

Lemma 15. $\mathsf{EQ}(x, \mathsf{compl}\, \mathsf{compl}\, x)$.

Proof. To show $\mathsf{P}(x, \mathsf{compl}\, \mathsf{compl}\, x)$, let r be connected to x but not to $\mathsf{compl}\, \mathsf{compl}\, x$. By R4a r is a non-tangential proper part of $\mathsf{compl}\, x$, so there is no region externally connected both to r and to $\mathsf{compl}\, x$. In particular r is not externally connected to x, since $\mathsf{EC}(x, \mathsf{compl}\, x)$ follows from Lemma 10, so $\mathsf{O}(r, x)$. But r is a part of $\mathsf{compl}\, x$ which is a contradiction by Lemma 10.

To show that $\mathsf{P}(\mathsf{compl}\, \mathsf{compl}\, x, x)$, suppose not and derive a contradiction using R4b and Lemma 10. □

Lemma 16. *If the conditions* $\mathsf{P}(w, \mathsf{sum}(x, y))$ *and* $\mathsf{O}(w, \mathsf{compl}\, x)$ *are satisfied then* $\mathsf{P}(\mathsf{prod}(w, \mathsf{compl}\, x), y)$.

Proof. Suppose, to the contrary, that $\mathsf{P}(w, \mathsf{sum}(x, y))$ and $\mathsf{O}(w, \mathsf{compl}\, x)$, but that $\mathsf{prod}(w, \mathsf{compl}\, x)$ is not a part of y. By R4b there is a region, say r, which is a part both of $\mathsf{prod}(w, \mathsf{compl}\, x)$ and of $\mathsf{compl}\, y$. Let s be a non-tangential proper part of r. By Lemma 14, s is a non-tangential proper part both of $\mathsf{compl}\, x$ and of $\mathsf{compl}\, y$. Thus by R4a and Lemma 15, s is not connected either to x or to y. This is a contradiction because s is a part of $\mathsf{sum}(x, y)$. □

We now arrive at the main result we need to establish distributivity.

Theorem 17. $\mathsf{P}(\mathsf{prod}(\mathsf{sum}(x, y), \mathsf{sum}(x, z)), \mathsf{sum}(x, \mathsf{prod}(y, z)))$.

Proof. Let r be connected to $\mathsf{prod}(\mathsf{sum}(x, y), \mathsf{sum}(x, z))$. We have to show that r is connected to $\mathsf{sum}(x, \mathsf{prod}(y, z))$. If r is connected to x, the conclusion follows immediately, so assume for the remainder of the proof that r is not connected to x. By R6 there is a region, w say, such that $\mathsf{P}(w, \mathsf{sum}(x, y))$, $\mathsf{P}(w, \mathsf{sum}(x, z))$, and $\mathsf{C}(w, r)$. Since r is not connected to x, $\mathsf{O}(w, \mathsf{compl}\, x)$ for if not, w would be a part of x and $\mathsf{C}(w, r)$ would imply $\mathsf{C}(r, x)$.

From Lemma 16 we deduce $\mathsf{P}(\mathsf{prod}(w, \mathsf{compl}\, x), y)$ and $\mathsf{P}(\mathsf{prod}(w, \mathsf{compl}\, x), z)$. So by Lemma 6

$$\mathsf{P}\big(\mathsf{prod}(w, \mathsf{compl}\, x), \mathsf{prod}(y, z)\big). \tag{1}$$

Now consider the two cases according to whether x overlaps w or not. If x does not overlap w, then $w = \mathsf{prod}(w, \mathsf{compl}\, x)$ by Lemma 11. Thus r is connected to $\mathsf{prod}(y, z)$, since it is connected to $\mathsf{prod}(w, \mathsf{compl}\, x)$, and we have (1) above. In the case that x does overlap w, r cannot be connected to $\mathsf{prod}(w, x)$, since r is not connected to x. But from Lemma 12 we have $w = \mathsf{sum}(\mathsf{prod}(w, x), \mathsf{prod}(w, \mathsf{compl}\, x))$, so r is connected to $\mathsf{prod}(w, \mathsf{compl}\, x))$, and hence by (1) above, to $\mathsf{prod}(y, z)$. □

From Theorem 17 it follows, by analysing cases where one of x, y or z is n or u, that $(x \vee y) \wedge (x \vee z) \leqslant x \vee (y \wedge z)$ holds in the lattice $\langle R \cup \{\mathsf{n}\}; \mathsf{n}, \mathsf{u}, \wedge, \vee\rangle$. By standard results of lattice theory [31, p. 38], this implies that the lattice is distributive.

The final stage in proving that the structure $\langle R \cup \{\mathsf{n}\}; \mathsf{C}, \mathsf{n}, \mathsf{u}, {}', \wedge, \vee\rangle$ is a BCA is to show that the equations $x' \wedge x = \mathsf{n}$ and $x' \vee x = \mathsf{u}$ hold, and that the four axioms A1–A4 for the connection relation, C, are satisfied. These details are all routine, and can easily be checked by the reader.

The constructions in Sections 4 and 3 define functions from the set of all strict models of RCC to the set of all BCAs and vice versa respectively. It is easy to see that these functions are inverses of each other so we have a bijection (or one-to-one correspondence) between the two sets. This result is important because it allows the wealth of existing mathematical results about Boolean algebras to be applied to RCC.

5. A construction for BCAs

This section introduces lattices with certain properties which can be used to construct models of RCC. A pseudocomplemented distributive lattice has a subset, the elements of which form a Boolean algebra. This result is Theorem 21 below. Provided the lattice itself satisfies two conditions, described in Section 5.2, we obtain a Boolean connection algebra, and hence a model of RCC. An important class of the pseudocomplemented distributive lattices are the relatively pseudocomplemented distributive lattices or Heyting algebras. It is not clear whether there are models of RCC which arise from pseudocomplemented distributive lattices but which do not come from Heyting algebras. However, working in the context of general pseudocomplemented distributive lattices is natural since the relative pseudocomplement operation in a Heyting algebra plays no part in the constructions. Another advantage is that it is usually simpler to verify that a specific structure is a pseudocomplemented distributive lattice, than it is to verify that it is a Heyting algebra.

5.1. Pseudocomplemented distributive lattices

Definition 18. In a lattice, A, a *pseudocomplement* of $a \in A$ is an element $m \in A$ such that for all x in A, $a \wedge x = \bot$ iff $x \leqslant m$.

Note that m is a pseudocomplement of a iff m is the greatest element of $\{x \in A \mid a \wedge x = \bot\}$. Thus a need not have a pseudocomplement, but if it does then the pseudocomplement is unique. A *pseudocomplemented distributive lattice* is defined as a distributive lattice, A, equipped with a unary operation $^* : A \to A$, such that, for all $a \in A$, a^* is the pseudocomplement of a. The following theorem contains some well-known properties which we need.

Theorem 19. *The following equations hold in any pseudocomplemented distributive lattice.*

1. $(x \vee y)^* = x^* \wedge y^*$,
2. $(x \wedge y)^* = (x^* \vee y^*)^{**}$,
3. $(x \wedge y)^{**} = x^{**} \wedge y^{**}$,
4. $(x \vee y)^{**} = (x^{**} \vee y^{**})^{**}$,
5. $\bot^* = \top$,
6. $\top^* = \bot$,
7. $x^{***} = x^*$,
8. $x \vee x^{**} = x^{**}$ (*i.e.* $x \leqslant x^{**}$),
9. $x \wedge x^* = \bot$.

It is possible to give a purely equational definition of pseudocomplemented distributive lattices. A proof that a distributive lattice equipped with a unary operation satisfying equations 1, 3, 5, 8, and 9 is a pseudocomplemented distributive lattice can be found in Lee's paper [37].

In a pseudocomplemented distributive lattice two subsets are of particular importance.

Definition 20. Let A be a pseudocomplemented distributive lattice. The set of *skeletal* elements of A is defined by $\mathcal{S}(A) = \{a \in A \mid a^{**} = a\}$. The set of *dense* elements of A is $\mathcal{D}(A) = \{a \in A \mid a^{**} = \top\}$.

It is well known [31, p. 150] that since $a = a^{**} \wedge (a \vee a^*)$ every $a \in A$ can be written as $a = b \wedge c$ where $b \in \mathcal{S}(A)$ and $c \in \mathcal{D}(A)$. The skeletal elements have been given many other names, including 'regular' [34, p. 10], and, since the assignment $a \mapsto a^{**}$ is a closure operation, in the sense of [6, p. 129], 'closed'. The term 'center' for $\mathcal{S}(A)$ is used by Priestley [51, p. 217]. Alternative descriptions of the two sets are given by

$$\mathcal{S}(A) = \{a^* \in A \mid a \in A\}, \qquad \mathcal{D}(A) = \{a \in A \mid a^* = \bot\}.$$

The following standard result plays a key role in the construction of a Boolean connection algebra from a suitable pseudocomplemented distributive lattice. A proof of the theorem can be found in [2, p. 157].

Theorem 21. *$\mathcal{S}(A)$ is a Boolean algebra where $\bot$ and $\top$ are as in A, the complement is the restriction of the pseudocomplement to $\mathcal{S}(A)$, and where the meet, $\sqcap$, and the join, $\sqcup$, are defined by $x \sqcap y = x \wedge y$, and $x \sqcup y = (x \vee y)^{**}$.*

Many important examples of pseudocomplemented distributive lattices support a binary operation called a relative pseudocomplement, which generalizes the concept of pseudocomplement. In a lattice, A, with elements a and b, the pseudocomplement of a relative to b is an element $a * b$ such that for all $x \in A$, $x \wedge a \leqslant b$ iff $x \leqslant a * b$. Relative pseudocomplements are unique when they exist. Pseudocomplements are a special case of relative pseudocomplements in that a pseudocomplement of a is a pseudocomplement of a relative to $\bot$. A distributive lattice with a binary operation assigning to any pair of elements a and b the pseudocomplement of a relative to b is called a Heyting algebra. One example of a Heyting algebra is the algebra of subgraphs of a graph which is used in Section 6 below. Another significant example for the present paper is the set of all open sets of any topological space.

5.2. Connectedness and inexhaustibility

We shall see below that, provided two conditions are satisfied, the Boolean algebra of skeletal elements of a pseudocomplemented distributive lattice is a Boolean connection algebra. The purpose of this section is to introduce and discuss these two conditions: connectedness and inexhaustibility.

Definition 22. A lattice, A, is *connected* if it does not contain elements $a \neq \bot$ and $b \neq \bot$ such that $a \vee b = \top$ and $a \wedge b = \bot$.

Thus A is connected if whenever $a \vee b = \top$ and $a \wedge b = \bot$ then $a = \bot$ or $b = \bot$. It is sometimes convenient to use the equivalent condition that whenever $a \vee b = \top$ and $a \wedge b = \bot$ then $a = \top$ or $b = \top$. The equivalence of these conditions follows from the facts that if $a \vee b = \top$ and $a = \bot$ then $b = \top$, and if $a \wedge b = \bot$ and $a = \top$ then $b = \bot$.

If A is the lattice of open sets of a topological space, X, then the above definition is the usual topological notion of connectedness, viz. X cannot be expressed as the union of two disjoint non-empty open sets. If A is a Boolean algebra, then A cannot be connected unless it has only one or two elements.

In order to show that $\mathcal{S}(A)$ gives rise to a Boolean connection algebra, we only need the condition that $\mathcal{S}(A)$, rather than A itself, does not contain non-zero elements a and b such that $a \vee b = \top$ and $a \wedge b = \bot$. However, this condition is no weaker than A being connected, as the following result shows.

Lemma 23. *A pseudocomplemented distributive lattice, A is connected if and only if there do not exist $x, y \in \mathcal{S}(A)$ such that $x \neq \bot$, $y \neq \bot$, $x \vee y = \top$ and $x \wedge y = \bot$.*

Proof. Trivially connectedness implies the condition. For the converse, suppose that the condition holds, and that $a \wedge b = \bot$ and $a \vee b = \top$ where $a, b \in A$. Using Theorem 19(3) we get $a^{**} \wedge b^{**} = \bot$, and since the operation * is expanding by Theorem 19(8), we see that $\top = a \vee b \leqslant a^{**} \vee b^{**}$. Since $a^{**}, b^{**} \in \mathcal{S}(A)$, we have that one of a^{**} and b^{**} is $\bot$, and thus one of a and b is $\bot$. □

In a Boolean connection algebra the relations $\leqslant$ and $<$ in the Boolean algebra correspond exactly to the relations P or 'part of', and PP or 'proper part of' in the corresponding model of the region-connection calculus.

Besides the partial order $\leqslant$, a distributive lattice, A, carries a relation called 'well inside' by Johnstone [34, p. 80]. For $a, b \in A$, we use $a \lessdot b$ to denote that a is well inside b. This relation, when restricted to $\mathcal{S}(A)$, will correspond closely to the non-tangential proper part relation, NTPP, in the model of the region-connection calculus which we will construct from A. More precisely, when $b \neq \top$, the non-tangential proper parts of b are exactly those a such that $a \lessdot b$.

Definition 24. Let a, b be elements of a distributive lattice A. We say that a is *well inside* b (written $a \lessdot b$) if there exists $c \in A$ such that $c \wedge a = \bot$ and $c \vee b = \top$.

Any distributive lattice has the property that $a \lessdot b$ implies $a \leqslant b$. To see this, suppose $c \wedge a = \bot$ and $c \vee b = \top$. Then $a \wedge b = a$ since $a \wedge b = (a \wedge c) \vee (a \wedge b) = a \wedge (c \vee b) = a$.

Lemma 25. *In a pseudocomplemented distributive lattice, $a \lessdot b$ iff $a^* \vee b = \top$.*

Proof. If $a^* \vee b = \top$, we get $a \lessdot b$ by taking $c = a^*$. Conversely, if $c \wedge a = \bot$, then $c \leqslant a^*$ so $a^* \vee b \geqslant c \vee b = \top$. □

In the special case of the Heyting algebra of open sets of a topological space, $H \lessdot G$ holds for open sets iff $\overline{H} \subseteq G$, where $\overline{H}$ is the closure of H.

Connectedness can be expressed in terms of the well inside relation. It follows easily from the definitions that A is connected iff the only elements $a \in A$ such that $a \lessdot a$ are $\bot$ and $\top$.

In order to construct a Boolean connection algebra out of a pseudocomplemented distributive lattice, A, we need that A is inexhaustible in the following sense.

Definition 26. A is *inexhaustible* if for every $b \in \mathcal{S}(A)$ there is some $a \in \mathcal{S}(A)$ such that $a \lessdot b$.

5.3. Relating inexhaustibility to regularity

Gotts showed [28] that the regular open sets of a regular, alias T_3, connected topological space provide a model of the region-connection calculus. We have already noted that, if A is the lattice of open sets of a topological space, X, then $\mathcal{S}(A)$ corresponds to the Boolean algebra of regular open sets of X. To understand how Theorem 30 below relates to Gotts' work, it is necessary to understand the relationship between a space X being regular, and the condition that its lattice of open sets is inexhaustible.

Various conflicting definitions of the terms regular and T_3 can be found in the literature. We will use the following definition of regular which has the same meaning as T_3 in [28].

Definition 27. A topological space, X, is *regular* if given any point $x \in X$ and open set, G, with $x \in G$, there is an open set H such that $x \in H$ and $\overline{H} \subseteq G$.

If we use A to denote the lattice of open sets of X, the regularity condition can be expressed as, for every $G \in A$, $G = \bigcup\{H \in A \mid \overline{H} \subseteq G\}$. Since $\overline{H} \subseteq G$ iff $H \lessdot G$ in A, the appropriate notion of regular for a distributive lattice is evident.

Definition 28. A distributive lattice, A, is *regular* if for every $a \in A$, a is the join of the set of elements well inside a,

$$a = \bigvee\{x \in A \mid x \lessdot a\}.$$

This definition does not require that A be complete, only that the set of elements well inside any element have a join. If A is a Boolean algebra, then A is regular since in this case the well inside relation is reflexive.

Lemma 29. *If A is regular then A is inexhaustible.*

Proof. If $a \in \mathcal{S}(A)$ is not $\bot$, then $\{x \in A \mid x \lessdot a\}$ must be non-empty, otherwise $a = \bigvee \varnothing = \bot$. Thus we can find x such that $x \lessdot a$. Now $x^{**} \in \mathcal{S}(A)$, and $x^{**} \lessdot a$ since $x^{**} \lessdot a$ iff $x^{***} \vee a = \top$ iff $x^{*} \vee a = \top$ iff $x \lessdot a$. $\square$

The converse of this result does not hold, as can be seen from the six element lattice in Fig. 1, which is inexhaustible but not regular. However, the lattice in Fig. 1 is not connected. In Section 6 we will present an example of a lattice which is connected, but which is inexhaustible and not regular.

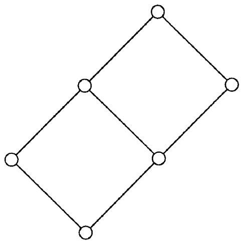

Fig. 1. A lattice which is inexhaustible but not regular.

5.4. The construction theorem

Theorem 30. *Let $A = \langle A; \bot, \top, {}^*, \vee, \wedge\rangle$ be a pseudocomplemented distributive lattice with $\langle \mathcal{S}(A); \bot, \top, {}', \sqcup, \sqcap\rangle$ as its Boolean algebra of skeletal elements, and let the relation* C *on $\mathcal{S}(A)$ be defined by $\mathsf{C}(x, y)$ iff $x^* \vee y^* \neq \top$. Suppose that A is connected and inexhaustible, and that $\mathcal{S}(A)$ contains more than two elements. Then $\langle \mathcal{S}(A); \mathsf{C}\rangle$ is a Boolean connection algebra.*

Proof. As before, R will denote $\mathcal{S}(A) - \{\bot\}$, and R_- will denote $R - \{\top\}$. Axiom A1 is easily checked, and A2 follows immediately from connectedness. The axiom A3 is equivalent to the condition $x^* \vee (y \sqcup z)^* = \top$ iff $x^* \vee y^* = \top$ and $x^* \vee z^* = \top$. A simple calculation shows that $x^* \vee (y \sqcup z)^* = (x^* \vee y^*) \wedge (x^* \vee z^*)$. The condition then holds since in any lattice $a \wedge b = \top$ if and only if both $a = \top$ and $b = \top$. The axiom A4 amounts to showing that given any $x \in R_-$ we can find $y \in R$ such that $x^* \vee y^* = \top$. This follows from the fact that A is inexhaustible and $x^* \in R_-$. □

By combining a special case of this theorem with Theorem 5, we obtain Gotts' result [28].

Corollary 31. *Let X be a connected, regular topological space, let R be the set of non-empty regular-open sets of X, and assume that R contains more than two elements. Define the relation C on R by $\mathsf{C}(H, K)$ iff $\overline{H} \cap \overline{K} \neq \varnothing$. Define $\mathsf{sum}(H, K)$ to be the interior of $\overline{H \cup K}$, define $\mathsf{prod}(H, K)$ to be $H \cap K$, and $\mathsf{compl}\, H$ to be the interior of $X - H$. Then $\langle R, \{\varnothing\}; X, \mathsf{compl}, \mathsf{sum}, \mathsf{prod}, \mathsf{C}\rangle$ is a model of the region-connection calculus.*

6. Example

In this section a general introduction to the structure of the set of all subgraphs of a graph is provided. Then, a specific example is examined: the binary subdivision graph. Finally, in Section 6.3, it is shown that the subgraphs of the binary subdivision graph lead to a pseudocomplemented distributive lattice which is inexhaustible and connected, but not regular. This gives an example of a construction of a model of RCC which does not arise from the results in [28] but which is covered by Theorem 30 above.

6.1. Subgraphs

One of the key examples of a Boolean algebra is the set of all subsets of a set. If we consider more generally the set of all subgraphs of a graph we obtain a Heyting algebra. The discussion below can easily be extended to directed graphs, but for our purposes it is sufficient to deal with only the undirected case.

An undirected graph $G = \langle G_N,\ G_A\rangle$ has a set, G_N, of nodes and a set, G_A, of arcs. Every arc $a \in G_A$ has a set of two end nodes, or just one in the case of a loop. There may be several arcs between two given nodes. The term 'multigraph' is sometimes used to refer to such a structure, and other authors [32, p. 10] use 'pseudograph'. However, our usage of

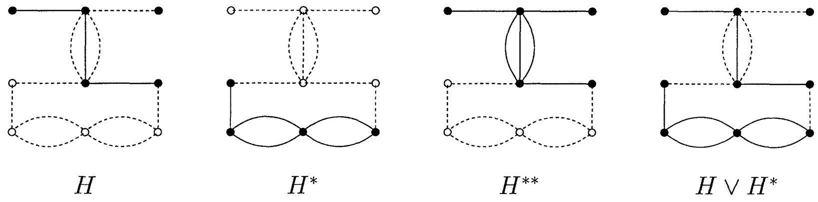

Fig. 2. Examples of operations on subgraphs.

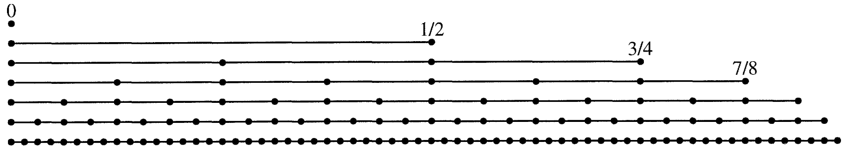

Fig. 3. Construction of the graph G.

'graph' is one of the standard ones [8, p. 176]. A set, A, of arcs determines a set of nodes, viz. those nodes which are an end of some arc in A, we will denote this set by $\mathsf{ends}\,A$. When A is a singleton, it is convenient to write $\mathsf{ends}\,a$ for $\mathsf{ends}\{a\}$. Dually, a set, N, of nodes determines a set $\mathsf{arcs}\,N$, containing all arcs with both ends in N. A subgraph, H, of a graph, G, is a graph $H = \langle H_N,\ H_A \rangle$, where $H_N \subseteq G_N$ and $H_A \subseteq G_A$, and where for every $a \in H_A$, $\mathsf{ends}\,a \subseteq H_N$. The symbol $\subseteq$ will be used for the subgraph relation as well as for subsets.

Given subgraphs, $H = \langle H_N,\ H_A \rangle$, and $K = \langle K_N,\ K_A \rangle$, their meet is $H \wedge K = \langle H_N \cap K_N,\ H_A \cap K_A \rangle$, and their join is $H \vee K = \langle H_N \cup K_N,\ H_A \cup K_A \rangle$. Relative pseudocomplement is defined as follows

$$(H * K)_N = H'_N \cup K_N \quad \text{and} \quad (H * K)_A = \big(H'_A \cup K_A\big) \cap \mathsf{arcs}\big(H'_N \cup K_N\big),$$

where H'_N is the set-theoretic complement of H_N in G_N. From the relative pseudocomplement, we obtain the pseudocomplement, $\langle H_N,\ H_A \rangle^* = \langle H'_N,\ \mathsf{arcs}(H'_N) \rangle$.

The diagrams in Fig. 2 show a subgraph, H, together with H^*, H^{**}, and $H \vee H^*$. In each diagram, the subgraph in question is highlighted by using solid arcs and nodes. Note that $H \vee H^* \neq \top$, and $H^{**} \neq H$.

6.2. Binary subdivision graph

Construct a graph G where the nodes are all rational numbers of the form $m/2^n$ where $0 \leqslant m < 2^n$, and n is a natural number (i.e., $0, 1, 2, 3, \ldots$). There is an arc between nodes a and b iff a and b can be expressed as $m/2^n$ and $(m+1)/2^n$. The graph can be visualized as the union of an infinite sequence of graphs, of which the first seven cases, corresponding to $n = 0, \ldots, 6$, are illustrated in Fig. 3. Each graph in the sequence includes all the nodes, but none of the edges, of the previous graph in the sequence. So in the union the nodes vertically aligned in the diagram are all identified.

In this example we can use $[a, b)$, where $0 \leqslant a \leqslant b \leqslant 1$, to denote the subgraph with no arcs, and containing only those nodes, x, of G for which $a \leqslant x < b$. In the Heyting algebra

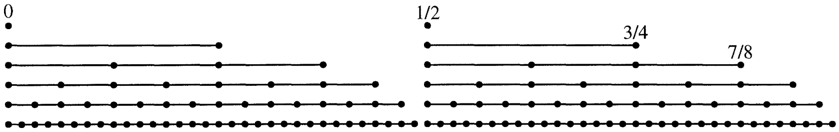

Fig. 4. Construction of the subgraph $[0, 1/2)^{**} \vee [1/2, 1)^{**}$ of G.

of all subgraphs of G, the subgraph having these nodes, and all arcs having both ends in $[a, b)$ is $[a, b)^{**}$.

Now consider all subgraphs of G which can be written as finite unions of the form $\bigcup_{i=1}^{k} [a_i, b_i)^{**}$. It is straightforward to check that these subgraphs form a subalgebra of the Heyting algebra of all subgraphs of G. To see that this Heyting algebra is not Boolean, consider, for instance, the subgraph $H = [0, 1/2)^{**}$. The pseudocomplement of this is $H^* = [1/2, 1)^{**}$. If we look at $H \vee H^*$ we find it contains all the nodes of the graph G, but for every n it lacks the arc joining $(2^{n-1} - 1)/2^n$ to $1/2$. This is illustrated in Fig. 4.

6.3. *Inexhaustibility and regularity*

The following example shows that even in the presence of connectedness, inexhaustibility does not imply regularity.

Consider the Heyting algebra of all subgraphs of a graph, G. If K and H are subgraphs such that $K \leqslant H$ then $K \lessdot H$ holds iff $\{a \in G_A \mid (\mathsf{ends}\, a) \cap K_N \neq \varnothing\} \subseteq H_A$, that is, every arc having an end in K must appear in H. The binary subdivision graph introduced above provides an example of a pseudocomplemented distributive lattice which is inexhaustible, but not regular. The elements of this lattice, L, are subgraphs of the form $\bigcup_{i=1}^{k} [a_i, b_i)^{**}$. The lattice is inexhaustible since every subgraph of the form $[a, a + d)^{**}$ has $[a + d/4, a + d/2)^{**}$ well inside it. Now consider the subgraph $H = [1/4, 1/2)^{**}$. First note that H cannot be the join of all the elements of the lattice well inside it. As the node $3/8$ lies in no element well inside H, no arc connected to this node lies in any K where $K \lessdot H$. Hence $[1/4, 3/8)^{**} \vee [3/8, 1/2)^{**}$ is an upper bound for $W = \{K \in L \mid K \lessdot H\}$, which is strictly less than H.

While this suffices to show that L is not regular, it is not much harder to see that the set W has no least upper bound. Suppose that $X = \bigcup_{i=1}^{k} [a_i, b_i)^{**}$ is a least upper bound for the set. Since every subgraph of the form $[1/4 + 1/2^n, 3/8)^{**}$ is well inside H when $n \geqslant 4$, we must have $a_i = 1/4$ for some i. Furthermore, there is no loss of generality in assuming this holds for a unique i. Now we can find d such that there is an arc with ends $1/4$ and $1/4 + d$ in the subgraph $[1/4, b_i)^{**}$. By replacing $[1/4, b_i)^{**}$ by $[1/4, 1/4 + d)^{**} \vee [1/4 + d/2, b_i)^{**}$ in the expression for X, we obtain a new subgraph Y. This new subgraph is strictly less than X, since it lacks the arc from $1/4$ to $1/4 + d$. However, Y has the same nodes as X, and the only arcs present in X but not in Y are some which are connected to $1/4$. To see this, note that an arc in X but not in Y must have length greater than $d/2$. One end of the arc must lie in $[1/4, 1/4 + d/2)^{**}$ so the distance of this node from $1/4$ must be less than $d/2$. It is easily seen that this distance cannot be non-zero. Hence Y is an upper bound for W, and $Y < X$, contradicting the assumption that X was the least upper bound.

7. A dual construction for BCAs

In addition to the construction given in Corollary 31 above, a model of the region-connection calculus can also be obtained from the non-empty regular closed sets of a connected regular topological space. This result is also to be found in Gotts' report [28]. To understand how this result relates to Theorem 30, we need to introduce the notion of duality for lattices.

If A is a partially ordered set, with partial order $\leqslant$, the *dual* of A, denoted A^{op}, is the partially ordered set having the same elements as A, but ordered by $\leqslant^{\mathrm{op}}$, where $x \leqslant^{\mathrm{op}} y$ iff $y \leqslant x$. Since a lattice, A, can be described in terms of its partial order, we can consider its dual, A^{op}. It can be shown that A^{op} will satisfy the axioms for a lattice. In A^{op} the meet is the join in A and vice versa, analogously the elements $\top$ and $\bot$ in A^{op} are the elements $\bot$ and $\top$ respectively in A. Beware that 'duality' has other meanings in the context of lattices: in particular it can mean topological spaces which are dual to lattices in a different sense.

Distributive lattices and Boolean algebras, like lattices, are self dual, in the sense that the dual of one of these structures is again a structure of the same kind. In the case of a Boolean algebra, A, the complement in A^{op} is the same operation on the elements of the algebra as the complement in A. In fact, the mapping $x \mapsto x'$ provides an isomorphism between a Boolean algebra and its dual. However, for arbitrary lattices A and A^{op} may be structurally very different.

Pseudocomplemented distributive lattices, and more specifically Heyting algebras, are not self dual, although all finite examples do have this property. The duals of these structures, described directly below, are called pseudosupplemented distributive lattices and co-Heyting algebras respectively. Although every pseudosupplemented distributive lattice arises as the dual of a pseudocomplemented distributive lattice, the study of these structures in their own right is often more natural. It is also useful where the same partially ordered set carries both kinds of structure simultaneously.

Definition 32. In a lattice, A, a *pseudosupplement* of $a \in A$ is an element $m \in A$ such that for all x in A, $a \vee x = \top$ iff $m \leqslant x$.

Note that m is a pseudosupplement of a iff m is the least element of $\{x \in A \mid a \vee x = \top\}$. Thus a need not have a pseudosupplement, but if it does then the pseudosupplement is unique. A *pseudosupplemented distributive lattice* is defined as a distributive lattice equipped with an operation ${}^\circ : A \to A$ such that for all $a \in A$, a° is the pseudosupplement of a.

Definition 33. Let A be a pseudosupplemented distributive lattice. The set of *central* elements of A is defined by $\mathcal{C}(A) = \{a \in A \mid a^{\circ\circ} = a\}$.

Theorem 34. *$\mathcal{C}(A)$ is a Boolean algebra where $\bot$ and $\top$ are as in A, the complement is the restriction of the pseudosupplement to $\mathcal{C}(A)$, and where the meet, $\sqcap$, and the join, $\sqcup$, are defined by $x \sqcap y = (x \wedge y)^{\circ\circ}$, and $x \sqcup y = x \vee y$.*

One special case of $\mathcal{C}(A)$ arises when A is the lattice of closed sets of a topological space, X. In this case $\mathcal{C}(A)$ is the set of regular closed sets of X.

In a pseudosupplemented distributive lattice, the well inside relation can be defined in terms of the $^\circ$ operation.

Lemma 35. *In a pseudosupplemented distributive lattice, $a \lessdot b$ iff $a \wedge b^\circ = \bot$.*

Definition 36. A is *inexhaustible* if for every $b \in \mathcal{C}(A)$ there is some $a \in \mathcal{C}(A)$ such that $a \lessdot b$.

If a pseudosupplemented distributive lattice A is inexhaustible then the pseudocomplemented distributive lattice A^{op} satisfies the condition that for any $b \in \mathcal{S}(A^{\mathsf{op}})$ there is some $a \in \mathcal{S}(A^{\mathsf{op}})$ such that $b \lessdot^{\mathsf{op}} a$. That is we can always find an a having b well inside it. This condition is equivalent to A^{op} itself being inexhaustible.

Lemma 37. *Let $A = \langle A; \bot, \top, {}^\circ, \vee, \wedge\rangle$ be a pseudosupplemented distributive lattice. Then A is inexhaustible iff the pseudocomplemented distributive lattice A^{op} is inexhaustible.*

Proof. Suppose that A is inexhaustible, and let $b \in \mathcal{S}(A^{\mathsf{op}})$. Then $b^* \in \mathcal{S}(A^{\mathsf{op}})$, so there is $a \in \mathcal{S}(A^{\mathsf{op}})$ such that $a \vee^{\mathsf{op}} b^{**} = \top^{\mathsf{op}}$, i.e., $a^{**} \vee^{\mathsf{op}} b = \top^{\mathsf{op}}$. Since $a^* \in \mathcal{S}(A^{\mathsf{op}})$, we conclude that A^{op} is inexhaustible. The converse is similar. □

It is straightforward to give a direct proof of Theorem 39, which appears below, by mimicking that for Theorem 30. However, by introducing the notion of the dual of a Boolean connection algebra, we obtain a much better understanding of how the two theorems are related.

Suppose $\langle A; \mathsf{C}\rangle$ is a structure consisting of a Boolean algebra A and a relation C on A. The structure need not be a Boolean connection algebra. The *dual* structure, denoted $\langle A; \mathsf{C}\rangle^{\mathsf{op}}$, is the structure $\langle A^{\mathsf{op}}; \mathsf{C}^{\mathsf{op}}\rangle$, where A^{op} is the Boolean algebra dual to A, and C^{op} is the relation on A^{op} defined by $\mathsf{C}^{\mathsf{op}}(x, y)$ iff $\mathsf{C}(x', y')$.

Theorem 38. *The structure $\langle A; \mathsf{C}\rangle$ is a Boolean connection algebra iff the dual structure $\langle A; \mathsf{C}\rangle^{\mathsf{op}}$ is a Boolean connection algebra.*

Proof. Routine calculation verifies that the axioms for a Boolean connection algebra in the structure imply that these axioms hold in the dual structure. □

Theorem 39. *Let $A = \langle A; \bot, \top, {}^\circ, \vee, \wedge\rangle$ be a pseudosupplemented distributive lattice with $\mathcal{C}(A)$ as its Boolean algebra of central elements, and let the relation C on $\mathcal{C}(A)$ be defined by $\mathsf{C}(x, y)$ iff $x \wedge y \neq \bot$. Suppose that A is connected and inexhaustible, and that $\mathcal{C}(A)$ contains more than two elements. Then the structure $\langle \mathcal{C}(A); \mathsf{C}\rangle$ is a Boolean connection algebra.*

Proof. The dual lattice, A^{op}, is a pseudocomplemented distributive lattice which is connected, and where $\mathcal{S}(A^{\mathsf{op}})$ has more than two elements. It is also inexhaustible by

Lemma 37. Thus A^{op} satisfies the hypotheses on the pseudocomplemented distributive lattice in Theorem 30, so the structure $\langle A; \mathsf{C}\rangle^{\mathsf{op}}$ is a Boolean connection algebra. Hence, by Theorem 38, $\langle A; \mathsf{C}\rangle$ is a Boolean connection algebra. □

By combining a special case of this theorem with Theorem 5, we obtain Gotts' result [28].

Corollary 40. *Let X be a connected, regular topological space, let R be the set of non-empty regular-closed sets of X, and assume that R contains more than two elements. Define the relation C on R by $\mathsf{C}(H, K)$ iff $H \cap K \neq \varnothing$. Also define $\mathsf{sum}(H, K)$ to be $H \cup K$, define $\mathsf{prod}(H, K)$ to the closure of the interior of $H \cap K$, and define $\mathsf{compl}\, H$ to be the closure of $X - H$. Then the structure $\langle R, \{\varnothing\}; X, \mathsf{compl}, \mathsf{sum}, \mathsf{prod}, \mathsf{C}\rangle$ is a model of the region-connection calculus.*

8. Conclusions and further work

This paper has demonstrated the value of a lattice theoretic approach to the study of spatial regions. Using such an approach, a description of a large class of models of RCC has been given in an entirely point-free manner. The work reported here has introduced an approach to constructing models of RCC which should be valuable in achieving a deeper understanding of the RCC formalism. Besides the construction of models, the paper has also introduced the concept of Boolean connection algebra. Such an algebra provides a neat separation of the mereological and topological aspects of a set of regions. It is likely to have applications beyond those found in this paper.

There are many possibilities for further research based on the ideas introduced in this paper. One relates to modelling spatial regions which are in some sense vague. The topic of regions with indeterminate boundaries has attracted much attention in the context of GIS, see [10] for a collection of recent papers. One approach to vagueness, which has not been fully exploited in GIS work, is the notion of a rough set. Rough sets were introduced in [46], more recent expositions of the theory and applications of rough sets are [47,48] and an overview is given in [18]. The subsets of a set form a Boolean algebra, the discovery of the algebraic structure of rough sets—regular double Stone algebras—is due to [15,33,49]. A recent brief introduction to these algebras and their connection with rough sets is [17] and another useful reference is [45]. By using this algebraic structure in place of a Boolean algebra, and developing suitably modified axioms for connection, it may be possible to develop a notion of rough connection algebra. There should be relationships with other approaches to vague regions, including some [14,29] which relate to extensions to RCC.

This kind of modification to the notion of Boolean connection algebra would entail investigation of a mereology of vague regions. Simons [57, p. 25] notes that non-classical mereology has not received much mathematical attention and contrasts this with case of non-classical propositional calculi as in [54]. Worboys has some structures related to rough sets in [70,71] and Düntsch et al. [20] have described generalizations of contact and 'part of' relations to the rough set case.

A second area for further work is the mereotopology of discrete space. The RCC axioms require that space is infinitely divisible, but discrete spaces are evidently important in implementations of spatial information systems, and their mereotopological aspects have only recently begun to be investigated [26,41]. It would thus be worthwhile investigating discrete analogues of the notion of BCA. Stell [61] has shown that some of Galton's work in [26] can usefully be expressed in terms of the algebraic structure of the set of subgraphs of a graph. In view of the example in Section 6 above, graphs should provide useful examples of models of spaces both in the atomic or discrete case as well as the non-atomic.

A third direction for extending the notion of BCA is to deal with spaces described at different levels of detail. The topic of multiresolution spatial data is of considerable importance in geographic information systems [42], and formal models of such data are being developed [23,63,64]. In this context it should be possible to develop a notion of a family of BCAs which vary over a lattice of levels of detail. This would be expected to provide an example of the stratified map spaces introduced in [63]. Both the issues of multiresolution and discrete spaces might be combined by considering graphs at different levels of detail [60,64].

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